CHAOTIC DISTRIBUTION OF NON-LINEAR SYSTEMS PERTURBED BY RANDOM NOISE

T. KAPITANIAK

Institute of Applied Mechanics, Technical University of Lodz, Stefanowskiego 1/15, 90-924 Lodz, Poland

Received 10 January 1986; accepted in revised form 26 April 1986

Chaotic behaviour of the distribution function of non-linear systems like the Duffing oscillator and the non-linear pendulum perturbed by random noise is reported.

It is well known that an anharmonic system with an external periodic perturbation

$$\ddot{x} + a\dot{x} + bx + cx^3 = B\cos\omega t \qquad (1)$$

can show chaotic behaviour for certain ranges of the parameters (see for example refs. [1–5]).

We consider the amplitude distribution function of the stochastic system

$$\ddot{x} + a\dot{x} + bx + cx^3 = B\cos\omega t + \eta(t), \qquad (2)$$

where $\eta(t)$ is the white noise with zero mean and correlation function $\langle\eta(t), \eta(t')\rangle = D\delta(t-t')$, $D < 1$ is constant, $\langle\ \rangle$ indicates the ensembly average to be studied.

If information is needed only about certain properties of the system without direct reference to the distribution function, eq. (2) may be numerically integrated by a Monte Carlo simulation method [6]. As we are mainly interested in the behaviour of the distribution function, in what follows we employ the method of the Fokker–Planck–Kolmogorov (FPK) equation.

To rewrite eq. (2) in the form for which the FPK equation can be easily written [6,7] the following transformation was made: $x = x_1$, $\dot{x} = x_2$, $x_3 = B\cos\omega t$, where x_3 is the solution of the following initial-value problem:

$$\dot{x}_3 = x_4, \quad \dot{x}_4 = -\omega^2 x_3, \qquad (3)$$

and $x_3(0) = B$, $x_4(0) = 0$. After the above transformation system (2) has the form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -ax_2 - bx_1 - cx_1^3 + x_3 + \eta(t),$$
$$\dot{x}_3 = x_4, \quad \dot{x}_4 = -\omega^2 x_3. \qquad (4)$$

With the distribution function denoted by $P(x_1, x_2, x_3, x_4, t \,|\, x_{10}, x_{20}, B, 0)$, where x_{10} and x_{20} are the initial conditions of the system (2) the Fokker–Planck–Kolmogorov equation satisfying eqs. (4) becomes

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x_1}[x_2 P] - \frac{\partial}{\partial x_2}\left[(-ax_2 - bx_1 - cx_1^3 + x_3)P\right] - \frac{\partial}{\partial x_3}[x_4 P] - \frac{\partial}{\partial x_4}\left[-\omega^2 x_3 P\right] + \frac{D}{2}\frac{\partial^2 P}{\partial x_2^2}. \qquad (5)$$

This equation was solved numerically by the path-integral method, which was exactly described in ref. [8]. The main advantage of this method is the efficiency in terms of computer time compared with other methods. To verify the path-integral calculations the same equation was solved also by finite-difference methods. The amplitude distribution function was calculated from the formula

$$P(x_1, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(x_1, x_2, x_3, x_4, t \,|\, x_{10}, x_{20}, B, 0)\, dx_2\, dx_3\, dx_4. \qquad (6)$$

For our numerical calculations we put in this example $a = 1.0$, $b = -10.0$, $c = 100.0$, $B = 0.1$. The initial conditions are deterministic, $x_{10} = 0.0$, $x_{20} = 0.0$ or $x_{10} = 0.1$, $x_{20} = 0.0$. The amplitude of the random noise was $D = 0.2$. Depending on the value of ω three types of distribution function can be obtained: (i) one-maximum curve $\omega = 3.2$ (fig. 1A), (ii) two-maxima curve $\omega = 3.4$ (fig. 1B) and (ii) multiple maxima chaotic curve $\omega = 3.5$ (fig. 1C). The multiple maxima curve corresponds to the value of parameters for which the system (1) shows chaotic behaviour. To characterize the chaotic behaviour of the distribution function the phase portraits were plotted for three characteristic types of the distribution functions (fig. 2).

Also the function $P(x_1, t)$ for a constant amplitude x_1 shows chaotic behaviour (fig. 3).

The chaotic type of distribution function is very sensitive to initial conditions (fig. 4).

As another example we consider the forced non-linear pendulum with external noise:

$$\ddot{x} + a\dot{x} + \lambda \sin x = B \cos \omega t + \eta(t), \quad (7)$$

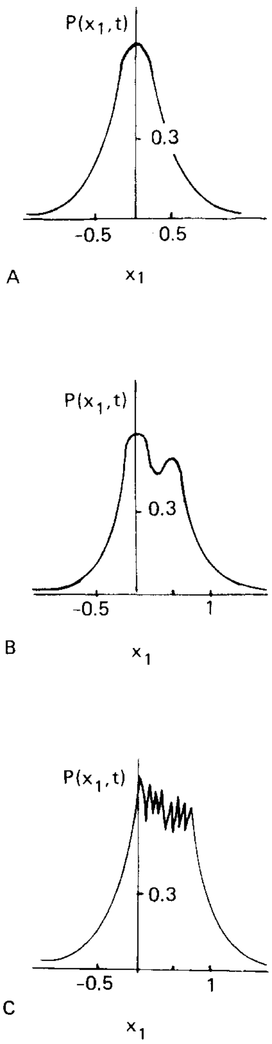

Fig. 1. Types of the amplitude distribution function for the anharmonic oscillator: $a = 1.0$, $b = -10.0$, $c = 100.0$, $B = 0.1$, $D = 0.2$, $t = 20.0$. (A) $\omega = 3.2$, (B) $\omega = 3.4$, (C) $\omega = 3.5$.

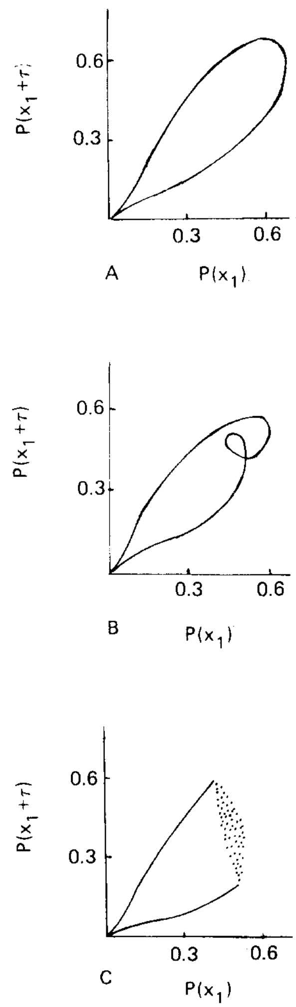

Fig. 2. Phase portraits of the amplitude distribution functions of fig. 1, $\tau = 0.2$.

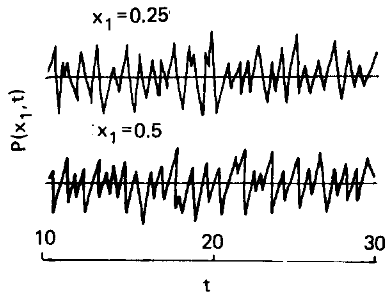

Fig. 3. $P(x_1, t)$ versus time for constant x_1.

where $\eta(t)$ is the white noise.

After the same transformation as for the anharmonic oscillator (2) the system (7) has the form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -ax_2 - \lambda \sin x_1 + x_3 + \eta(t),$$
$$\dot{x}_3 = x_4, \quad \dot{x}_4 = -\omega^2 x_3. \tag{8}$$

In this case the Fokker–Planck–Kolmogorov equation for the distribution function $P(x_1, x_2, x_3, x_4, t \,|\, x_{10}, x_{20}, B, 0)$ has the following form:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x_1}[x_2 P] - \frac{\partial}{\partial x_2}[(-ax_2 - \lambda \sin x_1 + x_3)P] - \frac{\partial}{\partial x_3}[x_4 P] - \frac{\partial}{\partial x_4}[-\omega^2 x_3 P] + \frac{D}{2}\frac{\partial^2 P}{\partial x_2^2}. \tag{9}$$

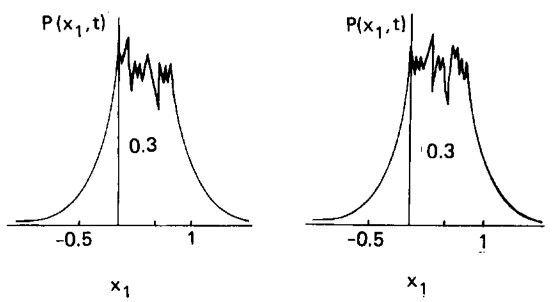

Fig. 4. Chaotic amplitude distribution function for different initial values: (left) $x_{10} = 0.0$, $x_{20} = 0.0$, (right) $x_{10} = 0.1$, $x_{20} = 0.0$.

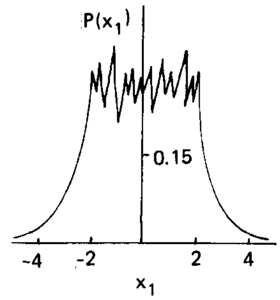

Fig. 5. Chaotic amplitude distribution function of the non-linear pendulum: $a = 1.0$, $b = 1.5$, $\lambda = 4.0$, $\omega = 0.25$, $D = 0.5$, $t = 20.0$, $x_{10} = 0$, $x_{20} = 0$.

In this example we put for numerical calculations $a = 1.0$, $b = 1.5$, $\lambda = 4.0$, $\omega = 0.25$, and the amplitude of the random noise is $D = 0.5$. The initial conditions are deterministic $x_{10} = 0.0$, $x_{20} = 0.0$ or $x_{10} = 0.0$, $x_{20} = 0.1$.

For these parameters the system (7) without random noise shows chaotic behaviour [9]. With random noise we obtain the chaotic amplitude distribution function (6) (fig. 5).

Also the stationary state distribution functions are chaotic (fig. 6).

To summarize, it seems that the chaotic distribution functions of a non-linear system perturbed by small external white noise are characteristic for all dynamic systems which show chaotic behaviour and may provide another description of chaos.

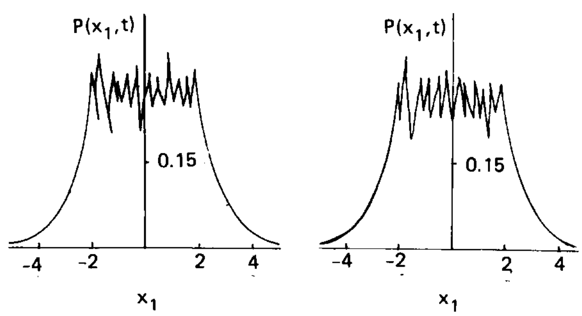

Fig. 6. Stationary state chaotic amplitude distribution functions: (left) $x_{10} = 0$, $x_{20} = 0$, (right) $x_{10} = 0$, $x_{20} = 0.1$.

References

[1] A. Kunick and W.-H. Steeb, J. Japan Phys. Soc. 54 (1985) 1220.

[2] W.-H. Steeb, W. Erig and A. Kunick, Phys. Lett. A 93 (1983) 267.

[3] B.A. Hubermann and J.P. Crutchfield, Phys. Rev. Lett. 43 (1979) 1743.

[4] Y. Ueda, Int. J. Non-linear Mech. 20 (1985) 481.

[5] S. Sato, M. Sano and Y. Sawada, Phys. Rev. A 28 (1983) 1654.

[6] D.L. Ermak and H. Buckholtz, J. Comp. Phys. 35 (1980) 169.

[7] I.I. Gichman and A.V. Skorchod, Stochastic differential equations (Springer, Berlin, 1972).

[8] G. Tagata, J. Sound Vibr. 58 (1978) 95.

[9] M.F. Wehner and W.G. Wolfer, Phys. Rev. A 27 (1983) 2663.

[10] D. D'Humieres, M. Beasley, B.A. Huberman and A. Libchaber, Phys. Rev. A 26 (1982) 3483.