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LETTER TO THE EDITOR

A PROPERTY OF A STOCHASTIC RESPONSE WITH BIFURCATION OF A NON-LINEAR SYSTEM

In the following an interesting analogy between the stochastic stationary response with bifurcation of a non-linear system perturbed by white noise and the bifurcation diagram of the unperturbed system is reported. For the chaotic movement of a non-linear system, the chaotic probability density functions have been obtained.

A stochastic process with bifurcation is understood to mean a random process given by a bifurcation solution of the non-linear Fokker-Planck-Kolmogorov equation [1, 2]. This type of probability density function (see Figure 1) is characterized by two maxima and tends to be intrinsically non-Gaussian.

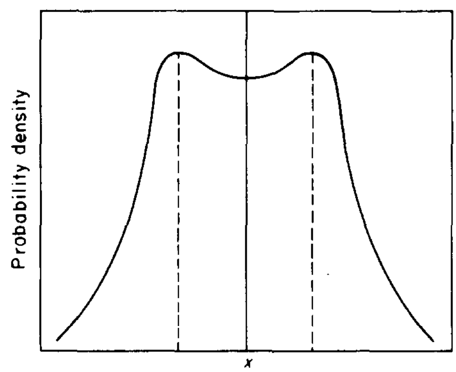

Figure 1. Probability density function of a stochastic process with bifurcation.

Consider the non-linear system given by the equation

$$(\mathrm{d}/\mathrm{d}t)\mathbf{f}(\mathbf{x}, t, \lambda) = 0, \tag{1}$$

where $\mathbf{x} = \mathrm{col}[x_1, \dots, x_n]$ is the vector of co-ordinates, $\mathbf{f} = \mathrm{col}[f_1, \dots, f_n]$, $t \in [0, \infty)$ is time and λ is the bifurcation parameter. The system is assumed to be perturbed by white noise, so that the equation of motion has the form

$$(\mathrm{d}/\mathrm{d}t)\mathbf{f}(\mathbf{x}, t, \lambda) = \mathbf{w}(t), \tag{2}$$

where $\mathbf{w}(t) = \mathrm{col}[w_1(t), \dots, w_n(t)]$, $w_i(t)$, $i = 1, \dots, n$, being a stochastic process with zero mean and the correlation function $\langle w_i(t) w_i(t')\rangle = D\delta(t - t')$, $\langle w_i(t) w_j(t)\rangle = 0$, $i = j$, in which $\langle \cdot \rangle$ indicates the ensemble average. The Fokker-Planck-Kolmogorov equation for the stationary state probability density function $P(x_1, \dots, x_n \,|\, x_{10}, \dots, x_{n0}, \lambda)$, where $x_{10}, \dots, x_{n0}$ are the initial conditions of the perturbed system, has the form

$$-(\partial/\partial x_i)[K_i - \tfrac{1}{2}(\partial/\partial x_j) Q_{ij}]P = 0, \tag{3}$$

where $K_i(\mathbf{x})$ is the drift vector and $\frac{1}{2}Q_{ij}(\mathbf{x})$ is the diffusion tensor; repeated indices imply a summation.

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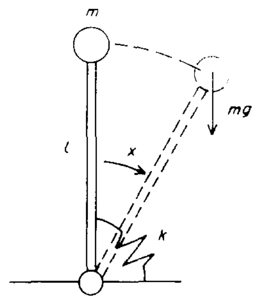

Figure 2. Model of the system. k, torsional stiffness; m, mass; l, length of the massless beam.

The probability density function of the system (2) can be calculated from the above equation by the path-integral method [3].

As a first example consider the system shown in Figure 2, for which

$$\dot{x}_1 = x_2, \dot{x}_2 = -ax_1 - \lambda \sin x_1, \tag{4}$$

where $x_1 = x$, $x_2 = \dot{x}$, $a = k/ml^2$ and $\lambda = P/ml$, for which the perturbed system is

$$\dot{x}_1 = x_2, \; \dot{x}_2 = -ax_1 - \lambda \sin x_1 + w(t), \tag{5}$$

where $w(t)$ is a white noise. The stationary state F-P-K equation for the system (5) has the form

$$-(\partial/\partial x_1)[x_2 P] - (\partial/\partial x_2)[(-ax_1 - \lambda \sin x_1)P] + (D/2)\partial^2 P/\partial x_2^2 = 0, \tag{6}$$

where $P(x_1, x_2 | x_{10}, x_{20}, \lambda)$ is the probability density function. The calculated amplitude probability density functions for different values of the bifurcation parameter λ,

$$P(x_1, \lambda) = \int_{-\infty}^{\infty} P(x_1, x_2, \lambda)\, dx_1,$$

are presented in Figure 3. The projection of the maxima of this function on the amplitude-bifurcation parameter plane gives the well known bifurcation diagram of the unperturbed system [4].

As another example consider the forced non-linear pendulum described by

$$\dot{x}_1 = x_2, \; \dot{x}_2 = -ax_2 - \lambda \sin x_1 + b \cos \omega t. \tag{7}$$

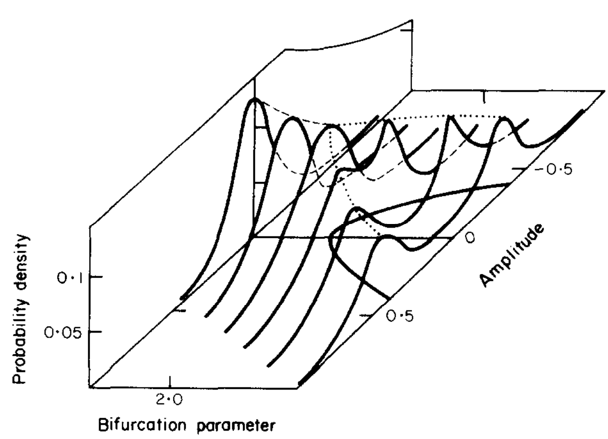

Figure 3. The analogy between stochastic bifurcation of the perturbed system and the bifurcation of the unperturbed system: $a = 2{\cdot}0$, $D = 0{\cdot}5$.

When one sets $x_3 = b\cos\omega t$ then x_3 is the solution of the initial value problem

$$\dot{x}_3 = x_4,\ \dot{x}_4 = -\omega^2 x, \tag{8}$$

$$x_3(0) = b, \qquad x_4(0) = 0, \tag{9}$$

so that the following equations are equivalent to equations (7) with the initial values (9):

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -ax_2 - \lambda\sin x_1 + x_3, \qquad \dot{x}_3 = x_4, \qquad \dot{x}_4 = -\omega^2 x_3. \tag{10}$$

The perturbed system has the form:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -ax_2 - \lambda\sin x_1 + x_3 + w(t),$$
$$\dot{x}_3 = x_4, \qquad \dot{x}_4 = -\omega^2 x_3, \tag{11}$$

where $w(t)$ is a white noise as before. The Fokker-Planck-Kolmogorov equation for the stationary state probability density function $P(x_1, x_2, x_3, x_4 | x_{10}, x_{20}, b, 0, \lambda)$ has the following form in this case:

$$-\frac{\partial}{\partial x_1}[x_2 P] - \frac{\partial}{\partial x_2}[(-ax_2 - \lambda\sin x_1 + x_3)P] - \frac{\partial}{\partial x_3}[x_4 P] - \frac{\partial}{\partial x_4}[-\omega^2 x_3 P] + \frac{D}{2}\frac{\partial^2 P}{\partial x_2^2}. \tag{12}$$

The amplitude probability density function is given by

$$P(x_1, \lambda) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(x_1, x_2, x_3, x_4 | x_{10}, x_{20}, b, 0, \lambda)\, dx_2\, dx_3\, dx_4.$$

If one chooses parameters such that the unperturbed system produces chaotic movement [5] one obtains the chaotic probability density functions shown in Figure 4.

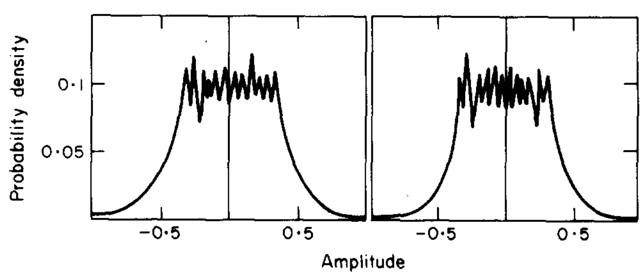

Figure 4. Examples of the chaotic probability density functions: $a = 1{\cdot}0$, $b = 1{\cdot}5$, $\lambda = 4{\cdot}0$, $\omega = 0{\cdot}25$, $D = 0{\cdot}5$, $A/x_{10} = 0$, $x_{20} = 0$, $B/x_{10} = 0$, $x_{20} = 0{\cdot}1$.

This property of the stationary state probability density function of the stochastic response of a non-linear system perturbed by white noise can be very useful for calculating the bifurcation diagrams and searching for the chaotic solutions of non-linear systems.

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