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NON-MARKOVIAN PARAMETRICAL VIBRATION

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Abstract—In this paper, the solution and its stability of Hill's equation with frequency and damping coefficient fluctuated by stochastic process modeled by Ornstein-Uhlenbeck noise is investigated. The application of Hermandez-Machado's and San Miguel's formulas allowed us to calculate the mean value and the second-order moments of the solution process. Non-Markovian effect of dependence of the parametrical vibration frequency on the noise intensity was determined. The investigations of the moments of the solution gave conditions for stability.

INTRODUCTION

UP TO NOW, any investigations of the stochastic parametrical vibration were based on the assumption that the vibration response is the Markovian process or can be approximated as such a process. In recent years, Hermandez-Machado's and San Miguel's investigations on the non-Markovian stochastic differential equations allow to consider the response of the parametrical system in the way which omits Markovian assumption.

In the present paper, we study the non-Markovian process defined by Hill's equation with frequency and damping coefficient fluctuated by non-white noise stochastic process.

As many of the physical processes can be modeled by the Ornstein-Uhlenbeck process [1], we take them into consideration.

The Ornstein-Uhlenbeck process is a Gaussian diffusion process with zero mean and correlation function (Feller [2])

$$\eta(t)\eta(t') = (D/\tau)\exp(-|t - t'|/\tau). \tag{1}$$

Let us consider the parametrical system described by Hill's eqn in the form:

$$\ddot{x} + (2h + \eta(t))\dot{x} + (\Omega^2 + \eta(t))x = 0. \tag{2}$$

Any solution of (2) is a non-Markovian one due to the fact that $\eta(t)$ is not white noise, also for every initial condition, the solution of (2) is a nonstationery process (Stratonovich [3]).

In the limit $\tau \rightarrow 0$ eqn (1) describes the correlation function of the white noise process and in this case the solution of (2) becomes Markovian.

In the following we study non-Markovian dynamical effect of the solution process and the stability properties of the system based on the eqns for the first- and the second-order moments.

After the transformation $x_1 = x$, $x_2 = \dot{x}$ eqn (2) has the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{bmatrix} 0 & -1 \\ \Omega^2 & 2h \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} \eta(t)x_1 \\ \eta(t)x_2 \end{pmatrix}. \tag{3}$$

MEAN VALUE EQUATIONS

Assuming that τ is a small parameter $\tau \ll 1$ we have the following Hermandez-Machado's–San Miguel's differential equation for the mean value [4]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x_i(t)\rangle = \sum_{j,k} (-A_{ij} + DB_{ik}B_{kj} - \tau DB_{ik}[A, B]_{kj})\langle x_i(t)\rangle + 0(\tau^2) \tag{4}$$

In our case we have

$$A = [A_{ij}] = \begin{bmatrix} 0 & -1 \\ \Omega^2 & 2h \end{bmatrix} \qquad B = [B_{ij}] = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \qquad AB - BA = [A, B_{ij}] = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

and the following equations for the mean value of the solution process:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x_1(t)\rangle = \langle x_2(t)\rangle,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x_2(t)\rangle = (-2h + D\tau)\langle x_2(t)\rangle + (-\Omega^2 + D\tau)\langle x_1(t)\rangle. \tag{5}$$

From eqns (5) we see that the mean value of the solution process oscillates with a frequency:

$$\omega = ((\Omega^2 - D\tau) - (h - \tfrac{1}{2}D\tau)^2)^{0.5}. \tag{6}$$

Equation (6) shows that the oscillation frequency of the system fluctuated by Ornstein-Uhlenbeck process is smaller than the frequency of the unfluctuated system.

This reduction of the oscillation frequency is a non-Markovian effect which cannot be reached by a Markovian approximation method. It disappears when $\tau \to 0$ as for white noise oscillation frequency is independent of the noise intensity D and

$$\omega = (\Omega^2 - h^2)^{0.5}.$$

The stability of the mean value can be investigated by the Ruth-Hurwitz method. The characteristic eqn of the system (5) has the following form:

$$\lambda^2 + \lambda(2h - D\tau) + (\Omega^2 - D\tau) = 0$$

so we obtained the following stability conditions:

$$D < 2h/\tau$$

and

$$D < \Omega^2/\tau.$$

THE SECOND-ORDER MOMENTS EQUATIONS

Hermandez-Machado's–San Miguel's differential equation for the second-order moments can be written in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x_i(t)x_j(t)\rangle = \sum_{k,l}(-A_{ik} + DB_{il}B_{lk} - \tau DB_{il}[A, B]_{lk})\langle x_k(t)x_j(t)\rangle + \sum_{k,l} D(B_{ik}B_{jl} - \tau B_{ik}[A, B]_{jl})\langle x_k(t)x_l(t)\rangle$$

and in case of the Hill's equation (2) has the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x_1^2\rangle = 2\langle x_2 x_1\rangle,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x_1 x_2\rangle = D(1 - \tau)\langle x_1^2\rangle + (-4h + D(3 - \tau))\langle x_2^2\rangle + (-2\Omega^2 + D(3 - \tau))\langle x_1 x_2\rangle,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x_2^2\rangle = (D(1 + 2\tau) - 2\Omega^2)\langle x_1^2\rangle + (-4h + D(1 - \tau))\langle x_1 x_2\rangle + D\tau\langle x_2^2\rangle. \tag{7}$$

For the given parameters D, τ eqn (7) is the linear system with constant coefficients which can be easily solved. The characteristic eqn for the system (7) has the form:

$$\lambda^3 + (2\Omega^2 - 3D)\lambda^2 + [D^2(-3 + 7\tau - 2\tau^2) + D(2 + 16h - \tau(2 + 2\Omega^2 + 8h)) - 16h^2]\lambda + [D^2(-3 - 6\tau + 3\tau^2) + D(4h + 6\Omega^2 + \tau(8h - 2\Omega^2) + -8h\Omega^2] = 0$$

and from Ruth-Hurwitz criterion we obtained the following conditions for stability of the second-order moments:

$$\tfrac{2}{3}\Omega^2 - D > 0,$$

$$D^3(3 - 7\tau + 2\tau^2) + D^2[(-3 + \Omega^2 + 8h) + \tau(20\Omega^2 - 24h + 12\tau + -3\tau^2)] + D[-2\Omega^2 + 32h\Omega^2 + 48h^2 - 4h - \tau(8h + 2\Omega^2 + 4\Omega^4 + -16h\Omega^2)] - 24\Omega^2 h > 0,$$

$$D^2(3 + 6\tau - 3\tau^2) + D(-4h - 8h\tau - 6\Omega^2 + 2\Omega^2\tau) + 8h\Omega^2 < 0.$$

These conditions give us the dependence of the noise parameters D and τ on the stability of the second-order moments of the solution.

The stationery-state value of the second-order moments:

$$\langle x_i x_j \rangle_{st} = \lim_{t \to \infty} \langle x_i x_j \rangle$$

is given by the stationery solution to (7):

$$\langle x_1 x_2 \rangle_{st} = 0,$$

$$D(1 - \tau)\langle x_1^2 \rangle_{st} + (-4h + D(3 - \tau))\langle x_2^2 \rangle_{st} = 0,$$

$$(D(1 - 2\tau) - 2\Omega^2)\langle x_1^2 \rangle_{st} + D\tau\langle x_2^2 \rangle_{st} = 0.$$

CONCLUSIONS

Applying the Hermandez-Machado's–San Miguel's eqns for the stochastic differential eqn with random coefficients modeled as the Ornstein-Uhlenbeck process, it is possible to investigate the parametrical vibration of the system driven by such a process. This way allows to omit the assumption that the response is the Markovian process or can be approximated as such a process.

Present investigation shows that the solution process has non-Markovian effect of dependence of the oscillation frequency on the noise intensity D. This effect disappears when $\tau \to 0$ and Ornstein-Uhlenbeck process becomes white noise.

The stability of the first and second-order moments of the solution process is given by a well-known condition of stability of the system of differential equation with constant coefficients.

The formulas given in the paper allow for the explicit calculation of the dependence of the mean value and second-order moments of the response on the parameters of the Ornstein-Uhlenbeck process and they are exact up to the second power of the small parameter τ.

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