Journal of Sound and Vibration (1987) 114(3), 588-592

QUANTIFYING CHAOS WITH AMPLITUDE PROBABILITY DENSITY FUNCTION

t is well known that the chaotic behaviour of a non-linear system can be described by 'arious quantities such as Lyapunov exponents, fractional dimension, phase portraits, ıutocorrelation functions, and Poincaré mapping. Their applications to systems with noise ıre not easy and they require very careful handling. In this paper another description of :haos based on the properties of the amplitude probability density function is developed.

As was shown in references [1-5], the shape of the amplitude probability density :unction of a non-linear oscillator, for example,

$$\ddot{x}+a\dot{x}+bx^3=B\cos\omega t, \tag{1}$$

depends on the character of the behaviour of the oscillator. For the harmonic solution described by the formula

$$x=A_i\cos(\omega t+\phi_i), \qquad i=1, \tag{2}$$

where A_i and ϕ_i ($i=1,2,3,4$) are constants, the shapes of the waveform and of the amplitude probability density function are like those in Figure 1(a). In Figure 1(b) the waveform and amplitude probability density function of the motion described by the 1/2

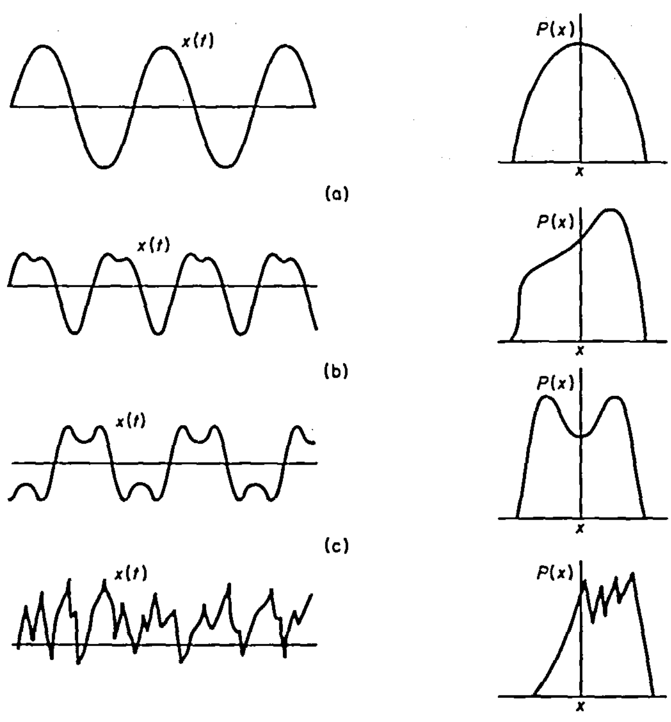

Figure 1. Examples of the waveforms and amplitude distributions of non-linear oscillations. (a) harmonic; (b) 1/2 harmonic; (c) first and third harmonic; (d) chaotic.

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subharmonic solution

$$x = A_2 \cos(\omega/2t + \phi_2) + A_3 \cos \omega t \tag{3}$$

are shown. For the case when the solution consists of the first and third harmonic,

$$x = A_1 \cos(\omega t + \phi_1) + A_4 \cos(3\omega t + \phi_4), \tag{4}$$

the shape of the amplitude probability density function is shown in Figure 1(c). It is characterized by the two maxima. Finally for chaotic behaviour of the oscillator the amplitude probability density function is described by the multi-maxima curve shown in Figure 1(d).

The multi-maxima curves occur due to the fact that in the oscillation waveform there exist values of the amplitude which are more probable than neighbouring values. For example, in Figure 2 the amplitudes a-a′ and c-c′ occur more often than the amplitude b-b′.

In references [2, 3] it was numerically shown that in the case of chaotic behaviour of the oscillator (1) the amplitude probability density function has multi-maxima but on the other hand a multi-maxima curve is not a sufficient indicator of chaotic behaviour.

A similar multi-maxima amplitude probability density function was obtained for the system (1) at $a = 0{\cdot}1$, $b = 1{\cdot}0$, $B = 10{\cdot}0$, and $\omega = 1{\cdot}03$, as is shown in Figure 3. In this case the system does not possess a chaotic behaviour but an almost periodic one, as shown on the Poincaré map in Figure 4.

The aim of the investigation reported in what follows is to provide an answer to the question of when the multi-maxima amplitude probability density function is a sufficient indicator for chaotic behaviour of the oscillator and to find a new indicator of chaos, based on the character of amplitude probability density function.

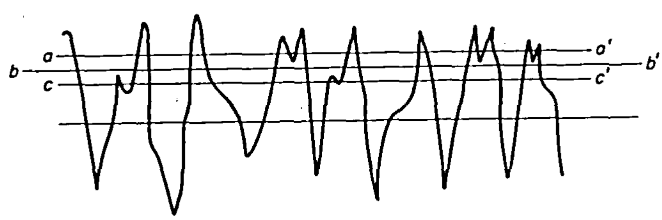

Figure 2. Explanation of multi-maxima amplitude distribution.

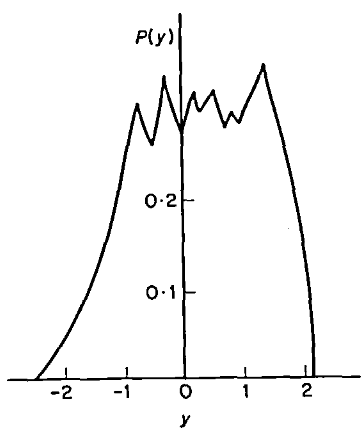

Figure 3. Amplitude distribution of the almost periodic behaviour of system (1) at $a = 0{\cdot}1$, $b = 1{\cdot}0$, $B = 10{\cdot}0$, $\omega = 1{\cdot}03$.

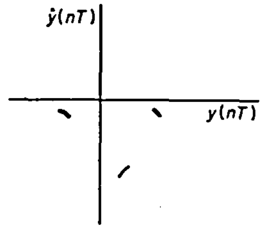

Figure 4. Poincaré map of the system of Figure 3.

It was observed that, for chaotic behaviour, the amplitude probability density function ɜpends on the length of the time history from which it is estimated, not only for the .itial period but also for large values of time. This property is illustrated in Figure 6 here, based on the time history of Figure 5 of the chaotic behaviour of the system 1 at $=0{\cdot}1$, $b=1{\cdot}0$, $B=10{\cdot}0$, and $\omega=1{\cdot}0$ [6], the amplitude probability density functions ere estimated with time period lengths of 80 s and 120 s, and different shapes of curves ere obtained.

The same property can be obtained from Fokker-Planck-Kolmogorov (FPK) equations quivalent to the system

$$\ddot{x}+a\dot{x}+bx^3=B\cos\omega t+\eta(t), \tag{5}$$

here $\eta(t)$ is a white noise random process with zero mean and correlation function $\eta(t)\eta(t')\rangle=K\delta(t-t')$, K is constant and $\langle\cdot\rangle$ indicates ensemble average.

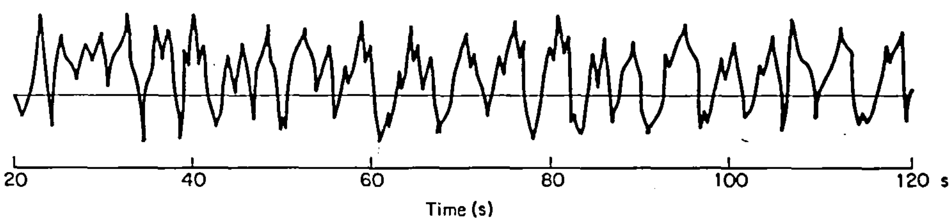

Figure 5. Time history of the chaotic behaviour of the system (1) at $a=0{\cdot}1$, $b=1{\cdot}0$, $B=10{\cdot}0$, $\omega=1{\cdot}0$.

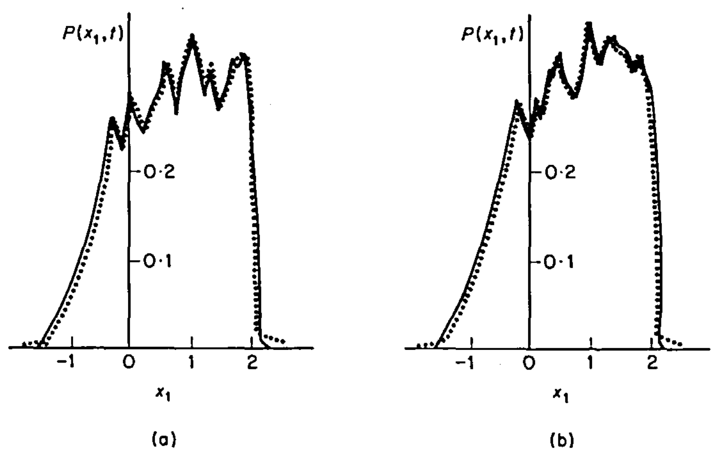

Figure 6. Amplitude distributions of the system of Figure 5. (a) $t=80{\cdot}0$; (b) $t=120{\cdot}0$.

According to references [1-4] the FPK equation for the joint probability density function $P(x_1, x_2, x_3, x_4, t|x_{10}, x_{20}, B, 0)$, where $x_1 = x$, $x_2 = \dot{x}$, $\dot{x}_3 = x_4$, $\dot{x}_4 = -\omega^2 x_3$, $x_3(0) = B$, $x_4(0) = 0$, and x_{10} and x_{20} are deterministic initial conditions of system (5), has the form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x_1}[x_2 P] + \frac{\partial}{\partial x_2}[(-ax_2 - bx_1^3 + x_3)P] + \frac{\partial}{\partial x_3}[x_4 P] + \frac{\partial}{\partial x_4}[-\omega^2 x_3 P] + \frac{K}{2}\frac{\partial^2 P}{\partial x_2^2}. \quad (6)$$

Equation (6) was solved numerically and the amplitude probability density function was calculated from the integral

$$P(x_{,1}, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(x_1, x_2, x_3, x_4, t|x_{10}, x_{20}, B, 0)\, dx_2\, dx_3\, dx_4. \quad (7)$$

The results obtained for the chaotic behaviour at $a = 0{\cdot}1$, $b = 1{\cdot}0$, $B = 10{\cdot}9$, $\omega = 1{\cdot}0$, $K = 0$, $t = 80{\cdot}0$ and $t = 120{\cdot}0$ are shown in Figure 6 as the dotted curves. These amplitude probability density functions also depend on time and they have the same shapes, respectively, as the ones obtained from time histories.

For comparison with this result, equation (6) was solved for the almost periodic system of Figure 3 and the results obtained for $t = 40{\cdot}0$, $80{\cdot}0$, $100{\cdot}0$, $120{\cdot}0$ are the same as in Figure 3.

To identify chaotic behaviour from the amplitude probability density function, it is necessary also to make a map of $P(x_1, t)$ versus $P(x_1, t+\tau)$ for constant x_1 and τ. Examples of such maps are shown in Figure 7. In the case of chaotic behaviour, the numerical results indicate that the maps have a Cantor set structure, Figure 7(a)-(c) (the parameters a, b, B, ω are the same as in Figure 5; $x_1 = 1{\cdot}0$, $0{\cdot}5$, $0{\cdot}0$ and $\tau = 5$).

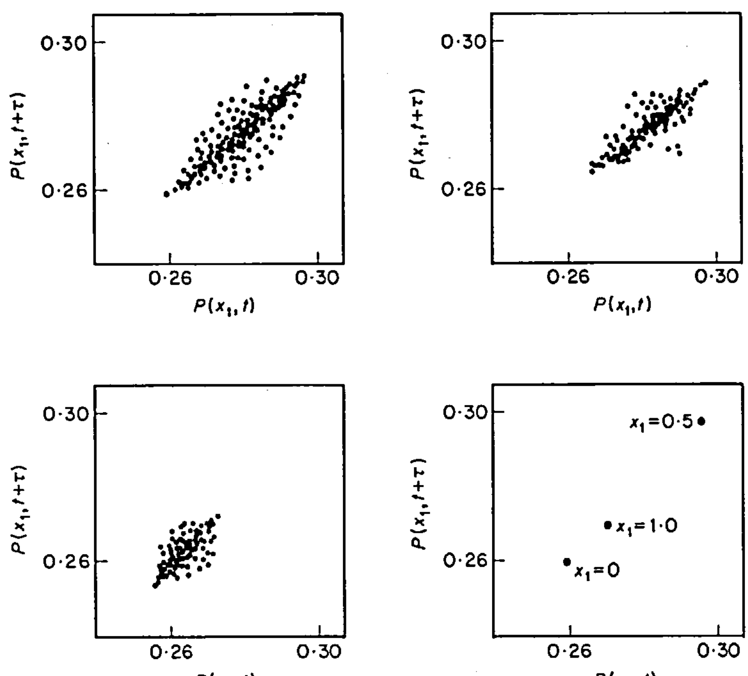

Figure 7. Maps $P(x_1, t)$ vs. $P(x_1, t+\tau)$ for constant x_1 and τ; a,b,c as for chaotic system of Figures 5 and 6 (a) $x_1 = 1{\cdot}0$; (b) $x_1 = 0{\cdot}5$; (c) $x_1 = 0$ $\tau = 5$; (d) almost periodic behaviour of Figure 3.

For regular motion the amplitude distribution does not depend on time and in this case the map $P(x_1, t)$ versus $P(x_1, t+\tau)$ consists of single point, as shown in Figure 7(d) (a, b, B, ω are the same as in Figure 3).

The character of the map $P(x_1, t)$ versus $P(x_1, t+\tau)$ provides the answer as to whether the system behaviour is chaotic or almost periodic. These maps may be new indicators of chaos.

To sum up, it seems that the character of the amplitude probability density function has distinctive features corresponding to whether the non-linear oscillator shows chaotic or regular behaviour. This indicator of chaos can be very useful for experimental investigations of non-linear systems as no real physical system is free of noise and its state of it can be specified only by the joint probability density function of the state coefficients. The amplitude probability density functions can be easily obtained from experimental data or from Monte-Carlo simulation, so this method can be used concurrently in calculations of other aspects of the chaotic behaviour.

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(*Received* 15 *December* 1986)

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