Journal of Sound and Vibration (1988) **121**(2), 259-268

COMBINED BIFURCATIONS AND TRANSITION TO CHAOS IN A NON-LINEAR OSCILLATOR WITH TWO EXTERNAL PERIODIC FORCES

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(*Received 4 July 1986, and in revised form 1 May 1987*)

The possibility of cascades of combined bifurcations is shown for a non-linear system with two external periodic forces. Some of these bifurcations have no analogy in the system with only one external force. The regions and stability of each bifurcation were estimated. In these regions examples of chaotic behaviour were found. For chaotic behaviour of the system, the amplitude probability density function was estimated and the chaotic shape of it was shown.

1. INTRODUCTION

The physical system with non-linear elastic characteristics and two external periodic forces,

$$\ddot{y} + h\dot{y} + y^3 = F_1 \cos \Omega_1 t + F_2 \cos \Omega_2 t, \tag{1}$$

where the driving frequencies Ω_1 and Ω_2 are incommensurable, is a generalization of Duffing's oscillator with one external force which has already had a wide literature devoted to it (see, for example, references [1-10]). For $\Omega_2 = 0$ system (1) is transformed to the system with periodic and constant external forces considered in reference [11].

If one puts $\Omega_1 = \Omega - \omega$ and $\Omega_2 = \Omega + \omega$, $F_1 = F_2 = F/2$, the system has the form

$$\ddot{y} + h\dot{y} + y^3 = F \cos \omega t \cos \Omega t. \tag{2}$$

System (2) can be interpreted as an ordinary forced Duffing's oscillator, $\ddot{y} + h\dot{y} + y^3 = B \cos \Omega t$, with a periodically perturbed amplitude of the external excitation: $B = F \cos \omega t$.

In the present paper the possibility of cascades of combined bifurcations, $(\Omega_1 \pm \Omega_2)/n$, $(\Omega_1 \pm 3\Omega_2)/n$, $(\Omega_2 \pm 3\Omega_1)/n$ and $3(\Omega_1 \pm \Omega_2)/n$, $n = 2, 4, \ldots$, will be shown.

The periods of these bifurcation solutions depend both on Ω_1 and Ω_2 and can be longer, or even shorter, in the first steps of the cascade than the periods of the external forces. On the higher steps of the cascade, the periods of all these solutions increase and tend to infinity. The period of all solutions on each step of the cascade is twice as long as on the previous step. This shows the possibility of the chaotic behaviour of the system (1).

For the chaotic solution of the system (1) the amplitude probability density function was estimated. The shape of this function is similar to those obtained via the Fokker-Planck-Kolmogorov (F-P-K) equation in references [12-14].

2. ALMOST PERIODIC SOLUTION

In the first approximation one considers the solution consisting of two components with periods of the exciting forces:

$$y(t) = C_1 \cos(\Omega_1 t + \vartheta_1) + C_2 \cos(\Omega_2 t + \vartheta_2). \tag{3}$$

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By substituting equation (3) into (1) and equating coefficients of $\cos(\Omega_1 t+\vartheta_1)$, $\sin(\Omega_1 t+\vartheta_1)$, $\cos(\Omega_2 t+\vartheta_2)$ and $\sin(\Omega_2 t+\vartheta_2)$ separately to zero, a set of algebraic equations for C_1, C_2, ϑ_1 and ϑ_2 is obtained:

$$\begin{aligned} -C_1\Omega_1^2+\tfrac{3}{4}C_1^3+\tfrac{3}{2}C_1C_2^2 &= F_1\cos\vartheta_1, & -hC_1\Omega_1 &= F_1\sin\vartheta_1,\\ -C_2\Omega_2^2+\tfrac{3}{4}C_2^3+\tfrac{3}{2}C_1^2C_2 &= F_2\cos\vartheta_2, & -hC_2\Omega_2 &= F_2\sin\vartheta_2. \end{aligned} \tag{4}$$

In addition to the first approximate solution (3) one can consider also a refined solution in the following form [1, 10, 15]:

$$y(t)=C_1\cos(\Omega_1 t+\vartheta_1)+C_2\cos(\Omega_2 t+\vartheta_2)+C_3\cos(3\Omega_1 t+\vartheta_3)+C_4\cos(3\Omega_2 t+\vartheta_4). \tag{5}$$

Neglecting the terms involving higher order harmonics and keeping only those terms linear in $C_{3,4}$, one can determine the amplitudes C_3, C_4 and phases ϑ_3, ϑ_4 to be as given by

$$\begin{aligned} -9C_3\Omega_1^2+\tfrac{3}{4}C_3^3+\tfrac{3}{2}C_3C_4^2 &= \tfrac{1}{4}C_1^3\cos(\vartheta_1-\vartheta_3), & -3hC_3\Omega_1 &= \tfrac{1}{4}C_1^3\sin(\vartheta_1-\vartheta_3),\\ -9C_4\Omega_2^2+\tfrac{3}{4}C_4^3+\tfrac{3}{2}C_3^2C_4 &= \tfrac{1}{4}C_2^3\cos(\vartheta_2-\vartheta_4), & -3hC_4\Omega_2 &= \tfrac{1}{4}C_2^3\sin(\vartheta_2-\vartheta_4). \end{aligned} \tag{6}$$

3. LOCAL STABILITY OF THE ALMOST PERIODIC SOLUTION (5)

To examine the local stability of the solution, equation (5) was perturbed by η, where $\eta \ll y$:

$$\tilde{y}(t)=y(t)+\eta(t). \tag{7}$$

After inserting expression (7) into equation (1) and taking into account equations (4) and (6) one obtains the following variational equation:

$$\ddot{\eta}(t)+h\dot{\eta}(t)+3y^2(t)\eta(t)+3y(t)\eta^2(t)+\eta^3(t)=0. \tag{8}$$

The local stability is examined by neglecting non-linear terms and considering the linear variational equation. Then inserting equation (5) gives

$$\begin{aligned} \ddot{\eta}(t)+h\dot{\eta}(t)+\eta(t)[&\lambda_0+\lambda_1\cos 2\Phi_1+\lambda_2\sin 2\Phi_1+\lambda_3\cos 2\Phi_2\\ &+\lambda_4\sin 2\Phi_2+\lambda_5\cos 2\Phi_3+\lambda_6\cos 2\Phi_4+\lambda_7\cos 4\Phi_1+\lambda_8\sin 4\Phi_1+\lambda_9\cos 4\Phi_2\\ &+\lambda_{10}\sin 4\Phi_2+\lambda_{11}\cos(\Phi_1+\Phi_2)+\lambda_{12}\cos(\Phi_1-\Phi_2)+\lambda_{13}\cos(\Phi_1+\Phi_4)\\ &+\lambda_{14}\cos(\Phi_1-\Phi_4)+\lambda_{15}\cos(\Phi_2+\Phi_3)+\lambda_{16}\cos(\Phi_2-\Phi_3)\\ &+\lambda_{17}\cos(\Phi_3+\Phi_4)+\lambda_{18}\cos(\Phi_3-\Phi_4)]=0, \end{aligned} \tag{9}$$

where

$$\begin{gathered} \lambda_0=\tfrac{1}{2}(C_1^2+C_2^2+C_3^2+C_4^2), \qquad \lambda_1=\tfrac{1}{2}C_1^2+C_1C_3\cos(\vartheta_3-2\vartheta_1),\\ \lambda_2=-C_1C_3\sin(\vartheta_3-2\vartheta_1), \qquad \lambda_3=\tfrac{1}{2}C_2^2+C_2C_4\cos(\vartheta_4-2\vartheta_2),\\ \lambda_4=-C_2C_4\sin(\vartheta_4-2\vartheta_2), \qquad \lambda_5=\tfrac{1}{2}C_3^2, \qquad \lambda_6=\tfrac{1}{2}C_4^2,\\ \lambda_7=C_1C_3\cos(\vartheta_3-2\vartheta_1), \qquad \lambda_8=\lambda_2, \qquad \lambda_9=C_2C_4\cos(\vartheta_4-2\vartheta_2),\\ \lambda_{10}=\lambda_4,\ \lambda_{11}=\lambda_{12}=C_2C_1, \qquad \lambda_{13}=\lambda_{14}=C_1C_4,\\ \lambda_{15}=\lambda_{16}=C_2C_3, \qquad \lambda_{17}=\lambda_{18}=C_3C_4,\\ \Phi_1=\Omega_1 t+\vartheta_1, \qquad \Phi_2=\Omega_2 t+\vartheta_2, \qquad \Phi_3=3\Omega_1 t+\vartheta_3, \qquad \Phi_4=3\Omega_2 t+\vartheta_4. \end{gathered} \tag{10}$$

In analogy to the classical Duffing's oscillator (1) the unstable regions which occur due to the terms $\cos 2\Omega_1$, $\sin 2\Omega_1$, $\cos 2\Omega_2$ and $\sin 2\Omega_2$ can be called the first order ones. There are two types of these regions; one of them takes place at $\Omega_1 \approx \sqrt{\lambda_0}$ and the other at $\Omega_2 \approx \sqrt{\lambda_0}$.

There are also two types of unstable regions of the third order, one at $\Omega_1 \approx \sqrt{\lambda_0}/3$ and the other at $\Omega_2 \approx \sqrt{\lambda_0}/3$.

These regions will not be considered in the present paper but particular attention will be given to the unstable regions which occur due to the terms $\cos(\Phi_1 \pm \Phi_2)$, $\cos(\Phi_1 \pm \Phi_4)$, $\cos(\Phi_2 \pm \Phi_3)$ and $\cos(\Phi_3 \pm \Phi_4)$, and which are called here combined regions. There are eight types of such regions. The first one occurs for $(\Omega_1+\Omega_2)/2 \approx \sqrt{\lambda_0}$, the second one for $(\Omega_1+\Omega_2)/2 \approx \sqrt{\lambda_0}$, the third one for $(\Omega_1+3\Omega_2)/2 \approx \sqrt{\lambda_0}$, the fourth one for $(\Omega_1 - 3\Omega_2)/2 \approx \sqrt{\lambda_0}$, the fifth one for $(3\Omega_1+\Omega_2)/2 \approx \sqrt{\lambda_0}$, the sixth one for $(3\Omega_1-\Omega_2)/2 \approx \sqrt{\lambda_0}$, the seventh one for $(3\Omega_1+\Omega_2)/2 \approx \sqrt{\lambda_0}$, and finally the eighth one for $3(\Omega_1-\Omega_2)/2 \approx \sqrt{\lambda_0}$.

4. COMBINED BIFURCATIONS

The first approximate solution in the combined unstable regions of these types according to Floquet theory is, for the first and second types,

$$\eta(t) = e^{\varepsilon_1 t} b^{(1)} \cos\left(\frac{\Omega_1+\Omega_2}{2} t + \vartheta^{(1)}\right), \qquad \eta(t) = e^{\varepsilon_2 t} b^{(2)} \cos\left(\frac{\Omega_1-\Omega_2}{2} t + \vartheta^{(2)}\right), \quad \text{(11a, b)}$$

for the third and fourth types,

$$\eta(t) = e^{\varepsilon_3 t} b^{(3)} \cos\left(\frac{\Omega_1+3\Omega_2}{2} t + \vartheta^{(3)}\right), \qquad \eta(t) = e^{\varepsilon_4 t} b^{(4)} \cos\left(\frac{\Omega_1-3\Omega_2}{2} t + \vartheta^{(4)}\right), \quad \text{(11c, d)}$$

for the fifth and sixth types,

$$\eta(t) = e^{\varepsilon_5 t} b^{(5)} \cos\left(\frac{3\Omega_1+\Omega_2}{2} t + \vartheta^{(5)}\right), \qquad \eta(t) = e^{\varepsilon_6 t} b^{(6)} \cos\left(\frac{3\Omega_1+\Omega_2}{2} t + \vartheta^{(6)}\right) \quad \text{(11e, f)}$$

and for the seventh and eighth types,

$$\eta(t) = e^{\varepsilon_7 t} b^{(7)} \cos\left(\frac{3(\Omega_1+\Omega_2)}{2} t + \vartheta^{(7)}\right), \qquad \eta(t) = e^{\varepsilon_8 t} b^{(8)} \cos\left(\frac{3(\Omega_1-\Omega_2)}{2} t + \vartheta^{(8)}\right). \quad \text{(11g, h)}$$

Here $\varepsilon_i > 0$ $i = 1, \ldots, 8$. At the stability limit of each type, $\varepsilon_i = 0$. To determine the boundaries of each type of the above unstable regions one inserts each of the solutions (11) into the variational equation (9) and then the conditions of a non-zero solution for b^i give the following criteria to be satisfied at the stability limit for each type. For the first (+) and second (−) types,

$$\left[-\left(\frac{\Omega_1 \pm \Omega_2}{2}\right)^2 + \lambda_0\right]^2 + \tfrac{1}{4}(\Omega_1 \pm \Omega_2)^2 h^2 - \tfrac{1}{4}\lambda_{11}^2 = 0, \quad \text{(12a)}$$

for the third (+) and fourth (−) types,

$$\left[-\left(\frac{\Omega_1 \pm 3\Omega_2}{2}\right)^2 + \lambda_0\right]^2 + \tfrac{1}{4}(\Omega_1 \pm 3\Omega_2)^2 h^2 - \tfrac{1}{4}\lambda_{13}^2 = 0, \quad \text{(12b)}$$

for the fifth (+) and sixth (−) types,

$$\left[-\left(\frac{3\Omega_1 \pm \Omega_2}{2}\right)^2 + \lambda_0\right]^2 + \tfrac{1}{4}(3\Omega_1 \pm \Omega_2)^2 h^2 - \tfrac{1}{4}\lambda_{15}^2 = 0, \quad \text{(12c)}$$

and for the seventh (+) and eighth (−) types,

$$[\tfrac{9}{4}(\Omega_1 \pm \Omega_2)^2 + \lambda_0]^2 + \tfrac{9}{4}(\Omega_1 \pm \Omega_2)^2 h^2 - \tfrac{1}{4}\lambda_{17}^2 = 0. \tag{12d}$$

To investigate the existence of combined bifurcations at the critical points

$$\Omega_1 \pm \Omega_2 = (\Omega_1 \pm \Omega_2)_{1,2} = x_{1,2}^{(1)(2)}, \qquad \Omega_1 \pm 3\Omega_2 = (\Omega_1 \pm 3\Omega_2)_{1,2} = x_{1,2}^{(3)(4)},$$

$$3\Omega_1 \pm \Omega_2 = (3\Omega_1 \pm \Omega_2)_{1,2} = x_{1,2}^{(5)(6)} \quad \text{and} \quad 3(\Omega_1 \pm \Omega_2) = [3(\Omega_1 \pm \Omega_2)]_{1,2} = x_{1,2}^{(7)(8)},$$

which are the roots of equations (12), the existence and stability of the steady state solution of the complete variational equation should be examined [16].

As the equations (12) have the same form as the similar equation obtained in reference [11], after carrying out the same calculations as in reference [11], one obtains the following formulae for the amplitudes of the bifurcation solutions. For the first (+) and second types (−),

$$b^{(1),(2)} = \frac{1}{3}\left[1 - \frac{h^2}{2\{\lambda_0 - \frac{1}{4}(\Omega_1 \pm \Omega_2)_1^2\}}\right][(\Omega_1 \pm \Omega_2) - (\Omega_1 \pm \Omega_2)_1],$$

or

$$b^{(1),(2)} = \frac{1}{3}\left[1 - \frac{h^2}{2\{-\lambda_0 + \frac{1}{4}(\Omega_1 \pm \Omega_2)_2^2\}}\right][(\Omega_1 \pm \Omega_2)_2 - (\Omega_1 \pm \Omega_2)], \tag{13a}$$

for the third (+) and fourth (−) types,

$$b^{(3),(4)} = \frac{1}{3}\left[1 - \frac{h^2}{2\{\lambda_0 - \frac{1}{4}(\Omega_1 \pm 3\Omega_2)_1^2\}}\right][(\Omega_1 \pm 3\Omega_2) - (\Omega_1 \pm 3\Omega_2)_1],$$

or

$$b^{(3),(4)} = \frac{1}{3}\left[1 - \frac{h^2}{2(\frac{1}{4}(\Omega_1 \pm 3\Omega_2)_2^2 - \lambda_0)}\right][(\Omega_1 \pm 3\Omega_2)_2 - (\Omega_1 \pm 3\Omega_2)], \tag{13b}$$

for the fifth (+) and sixth (−) types,

$$b^{(5),(6)} = \frac{1}{3}\left[1 - \frac{h^2}{2\{\lambda_0 - \frac{1}{4}(\Omega_2 \pm 3\Omega_1)_1^2\}}\right][(\Omega_2 \pm 3\Omega_1) - (\Omega_2 \pm 3\Omega_1)_1],$$

or

$$b^{(5),(6)} = \frac{1}{3}\left[1 - \frac{h^2}{2\{\frac{1}{4}(\Omega_2 \pm 3\Omega_1)_2^2 - \lambda_0\}}\right][(\Omega_2 \pm 3\Omega_1)_2 - (\Omega_2 \pm 3\Omega_1)], \tag{13c}$$

for the seventh (+) and eighth (−) types,

$$b^{(7),(8)} = \frac{1}{3}\left[1 - \frac{h^2}{2\{\lambda_0 - \frac{1}{4}(3\Omega_1 \pm 3\Omega_2)_1^2\}}\right][(3\Omega_1 \pm 3\Omega_2) - (3\Omega_1 \pm 3\Omega_2)_1],$$

or

$$b^{(7),(8)} = \frac{1}{3}\left[1 - \frac{h^2}{2\{\frac{1}{4}(3\Omega_1 \pm 3\Omega_2)_2^2 - \lambda_0\}}\right][(3\Omega_1 \pm 3\Omega_2)_2 - (3\Omega_1 \pm 3\Omega_2)]. \tag{13d}$$

By making use of the Routh–Hurwitz criterion, the following conditions for the stability of the solutions (11) for $\varepsilon_i = 0$ are also then obtained:

$$\frac{3}{2}\frac{b^{(1),(2)}}{(\Omega_1 \pm \Omega_2)^2}[\tfrac{1}{4}(\Omega_1 \pm \Omega_2)_{1,2}^2 - \lambda_0] > 0, \qquad \frac{3}{2}\frac{b^{(3),(4)}}{(\Omega_1 \pm 3\Omega_2)^2}[\tfrac{1}{4}(\Omega_1 \pm 3\Omega_2)_{1,2}^2 - \lambda_0] > 0,$$

$$\frac{3}{2}\frac{b^{(5),(6)}}{(3\Omega_1 \pm \Omega_2)^2}[\tfrac{1}{4}(3\Omega_1 \pm \Omega_2)_{1,2}^2 - \lambda_0] > 0, \qquad \frac{3}{2}\frac{b^{(7),(8)}}{9(\Omega_1 \pm \Omega_2)^2}[\tfrac{9}{4}(\Omega_1 \pm \Omega_2)_{1,2}^2 - \lambda_0] > 0. \tag{14}$$

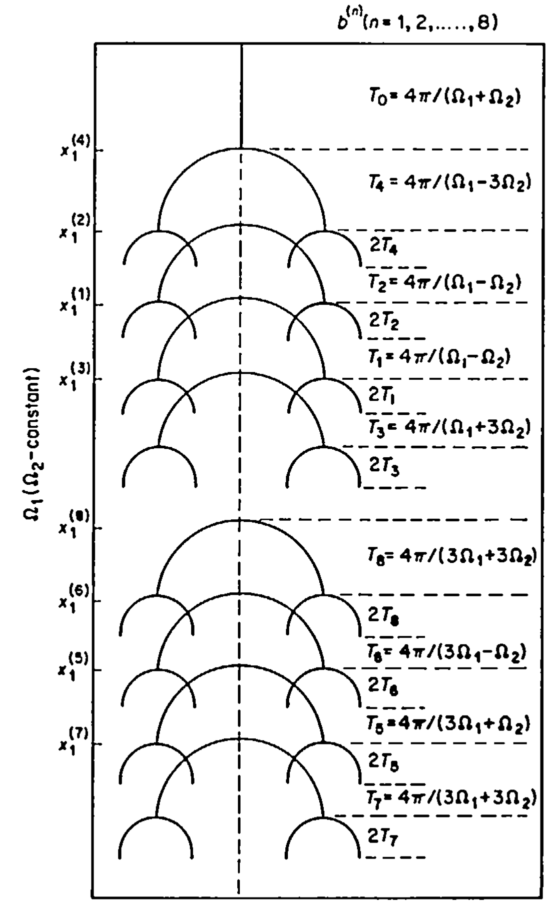

Figure 1. Scheme of the cascades of combined bifurcations.

From equations (13) one has that

$$\begin{aligned}
&\tfrac{1}{4}(\Omega_1 \pm \Omega_2)_1^2 - \lambda_0 < 0 \quad \text{and} \quad \tfrac{1}{4}(\Omega_1 \pm \Omega_2)_2^2 - \lambda_0 > 0,\\
&\tfrac{1}{4}(\Omega_1 \pm 3\Omega_2)_1^2 - \lambda_0 < 0 \quad \text{and} \quad \tfrac{1}{4}(\Omega_1 \pm \Omega_2)_2^2 - \lambda_0 > 0,\\
&\tfrac{1}{4}(3\Omega_1 \pm \Omega_2)_1^2 - \lambda_0 < 0 \quad \text{and} \quad \tfrac{1}{4}(3\Omega_1 \pm \Omega_2)_2^2 - \lambda_0 > 0,\\
&\tfrac{9}{4}(\Omega_1 \pm \Omega_2)_1^2 - \lambda_0 < 0 \quad \text{and} \quad \tfrac{9}{4}(\Omega_1 \pm \Omega_2)_2^2 - \lambda_0 > 0.
\end{aligned} \tag{15}$$

The above conditions show that all these bifurcation solutions (11) are stable in the neighbourhood of $x_2^{(j)}$, $j = 1, \ldots, 8$ and unstable in the neighbourhood of $x_1^{(j)}$.

By the same method the existence is shown of the further combined bifurcations with periods twice as long as a bifurcation of the appropriate type. Finally, one obtains the stable bifurcation diagram shown in Figure 1.

5. TRANSITION TO CHAOS

The formulas (12) provide the possibility of estimating the value of each amplitude and frequency at the stability limit of each type and from this to determine the frequencies

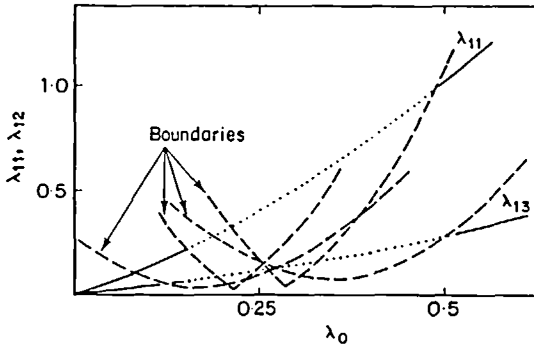

Figure 2. Loci of λ_0, λ_{11} and λ_{13} in equations (10) for varying $F_1 = F_2$; $\Omega_1 = 1{\cdot}2$, $\Omega_2 = 0{\cdot}2$, $h = 0{\cdot}1$.

for which the bifurcations take place. An example of such estimation for the lower $(\Omega_1 \pm \Omega_2)/2$ and $(\Omega_1 \pm 3\Omega_2)/2$ unstable regions and constant Ω_2 is shown in Figure 2. The parameters λ_0, λ_{11}, λ_{13} were calculated from equations (10) and the stability boundaries from equations (12).

The Ω_1 frequency zones (Ω_2 and other parameters are constant) for which the combined bifurcations of the first four types take place are shown in Figure 3. The region of the first type bifurcation $(\Omega_1 + \Omega_2)/2$ is shown between $x_1^{(1)}$ and $x_2^{(1)}$ and the region of the second type bifurcation $(\Omega_1 - \Omega_2)/2$ between $x_1^{(2)}$ and $x_2^{(2)}$. The third and the fourth type bifurcations $(\Omega_1 \pm 3\Omega_2)/2$ take place between $x_1^{(3)} - x_2^{(3)}$ and $x_1^{(4)} - x_2^{(4)}$.

In the zone between $x_1^{(4)}$ and $x_2^{(2)}$ all of these bifurcations exist and here chaotic behaviour is most likely. In Figure 4 Poincaré maps of the system (1) chaotic behaviour which were found at $h = 0{\cdot}1$, $\Omega_2 = 0{\cdot}2$, $F_1 = F_2 = 10{\cdot}0$ and $\Omega_1 = 1{\cdot}17$ (Figure 4(a)), and $\Omega_1 = 1{\cdot}20$ (Figure 4(b)) are presented.

In Figure 5 the frequency spectra (amplitudes of the Fourier components versus frequency for $y(t)$) of the above system are presented. First at $\Omega_1 = 1{\cdot}22$ the amplitude spectrum consists of the Ω_1, Ω_2, $3\Omega_1$ and $3\Omega_2$ periodic components and combined subharmonic components $(\Omega_1 + \Omega_2)/2$, $(\Omega_1 - \Omega_2)/2$ and $(\Omega_1 + 3\Omega_2)/2$, $(\Omega_1 - 3\Omega_2)/2$. Then at $\Omega_1 = 1{\cdot}20$ the amplitude spectrum becomes continuous in the neighbourhood of subharmonic components and the system shows chaotic behaviour. The chaotic behaviour still takes place at $\Omega_1 = 1{\cdot}17$. On further decrease of the frequency a strange attractor disappears and at $\Omega_1 = 1{\cdot}15$ the frequency spectrum consists of the periodic components Ω_1, Ω_2, $3\Omega_1$ and $3\Omega_2$.

Examples of the chaotic behaviour of a negative resistance non-linear oscillator are presented in reference [17].

6. AMPLITUDE PROBABILITY DENSITY FUNCTION OF THE CHAOTIC BEHAVIOUR

As the amplitude of the chaotic behaviour seems to be "random" the amplitude probability density function is worth estimating: see Figure 6. The shape of the probability density function is similar to those obtained in other papers [12–14], which were computed from the Fokker-Planck-Kolmogorov (F-P-K) equations equivalent to the chaotic system perturbed by random noise. In the present case this will be

$$\ddot{y} + h\dot{y} + y^3 = F_1 \cos \Omega_1 t + F_2 \cos \Omega_2 t + n(t), \tag{16}$$

where $n(t)$ is a white noise process with zero mean and correlation function $\langle n(t)n(t')\rangle = K\delta(t-t')$ where K is constant and $\langle\cdot\rangle$ indicates the ensemble average.

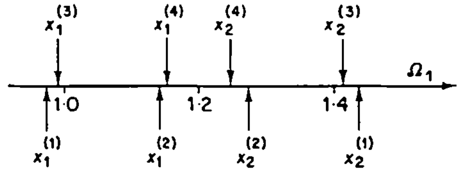

Figure 3. Ω_1 frequency zones of the combined bifurcations: the first type $x_1^1 - x_2^1$; the second type $x_1^2 - x_2^2$; the third type $x_1^3 - x_2^3$; the fourth type $x_1^4 - x_2^4$.

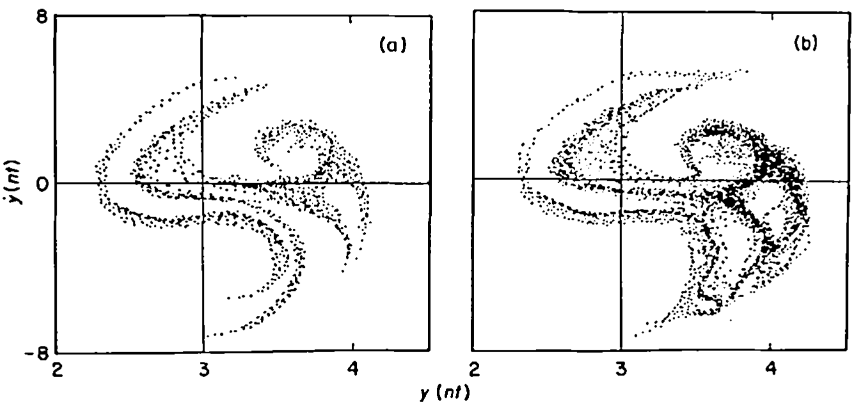

Figure 4. Poincaré maps of the chaotic behaviour; $T = 2\pi/\Omega_2$, $\Omega_2 = 0{\cdot}2$, $h = 0{\cdot}1$. (a) $\Omega_1 = 1{\cdot}17$, (*b*) $\Omega_1 = 1{\cdot}2$; $F_1 = F_2 = 10{\cdot}0$.

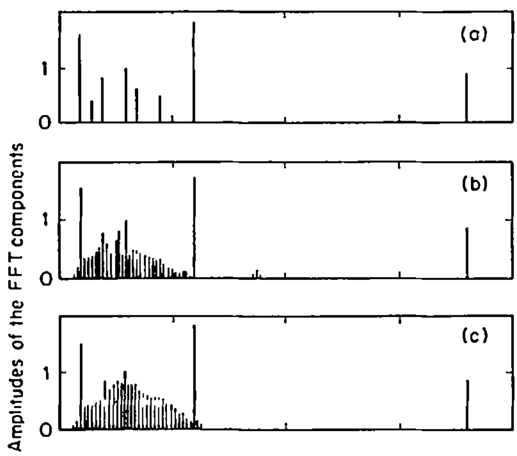

Figure 5. Frequency spectra; $\Omega_2 = 0{\cdot}2$, $h = 0{\cdot}1$, $F_1 = F_2 = 10{\cdot}0$, (a) $\Omega_1 = 1{\cdot}15$, (b) $\Omega_1 = 1{\cdot}17$, (*c*) $\Omega_1 = 1{\cdot}2$, (d) $\Omega_1 = 1{\cdot}22$.

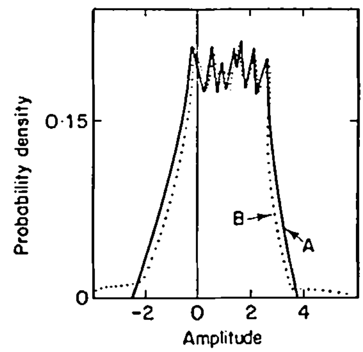

Figure 6. Amplitude probability density function of the chaotic behaviour of the system (1) for $t = 40{\cdot}0$. ——, Estimated from time history, $\cdots\cdots$, F-P-K equation.

After the transformation $y = x_1$, $y = x_2$, $x_3 = F_1 \cos \Omega_1 t$, $x_5 = F_2 \cos \Omega_2 t$, where x_3 and x_5 are the solutions of the initial value problems

$$\dot{x}_3 = x_4, \quad \dot{x}_4 = -\Omega_1^2 x_3 \quad \text{and} \quad \dot{x}_5 = x_6, \quad \dot{x}_6 = -\Omega_2^2 x_5,$$
$$x_3(0) = F_1, \quad x_4(0) = 0, \quad x_5(0) = F_2, \quad x_6(0) = 0,$$

one obtains the F-P-K equation for the joint probability density function $P(x_1, \ldots, x_6, t \mid x_{10}, x_{20}, F_1, 0, F_2, 0)$, where x_{10} and x_{20} are the initial conditions of the system (16), in the form

$$\partial P/\partial t = (\partial/\partial x_1)[x_2 P] + (\partial/\partial x_2)[(-hx_2 - x_1^3 + x_3 + x_5)P] + (\partial/\partial x_3)[x_4 P] + (\partial/\partial x_4)[-\Omega_1^2 x_3 P] + (\partial/\partial x_5)[x_6 P] + (\partial/\partial x_6)[-\Omega_2^2 x_5 P] + (K/2)\,\partial^2 P/\partial x_2^2. \quad (17)$$

The amplitude probability density function presented in Figure 6 was estimated according to the limited part of the time history (40 s) of the chaotic behaviour of the system (1) in the absence of random noise so as to compare this result with solution of the F-P-K equation (17), this being solved for $K = 0$ and $t = 40{\cdot}0$. Equation (17) was solved by a path-integral method [18] and the amplitude probability density function was obtained from the formula

$$P(x_1, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \times P(x_1, \ldots, x_6, t \mid x_{10}, x_{20}, F_1, 0, F_2, 0)\, \mathrm{d}x_2\, \mathrm{d}x_3\, \mathrm{d}x_4\, \mathrm{d}x_5\, \mathrm{d}x_6.$$

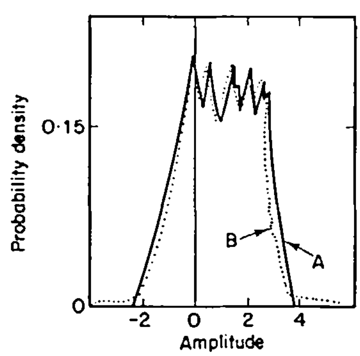

Figure 7. Amplitude probability density function of the chaotic behaviour for $t = 80{\cdot}0$. A, ——, Estimated from time history, B, $\cdots\cdots$, F-P-K equation.

Similar curves were obtained by these two methods: see Figure 6. For comparison, the amplitude probability density function for $t = 80{\cdot}0$, whose shape is different from the previous one, is presented in Figure 7.

7. CONCLUSIONS

Investigations of a non-linear oscillator with two external periodic forces show that combined bifurcations of periods $2n\pi/(\Omega_1 \pm \Omega_2)$, $2n\pi/(\Omega_1 \pm 3\Omega_2)$, $2n\pi/(3\Omega_1 \pm \Omega_2)$ and $2n\pi/3(\Omega_1 \pm \Omega_2)$, $n = 2, 4, \ldots$, can occur. Such bifurcations are characteristic for a system with two external forces and do not have an analogy in the system with only one force.

The system can show chaotic behaviour in a narrow zone of one frequency of the external force while the other is constant in the neighbourhood of the points where the losses of stability of combined bifurcations are predicted.

The chaotic zone is a transition zone between the almost periodic solution,

$$y(t) = C_1 \cos(\Omega_1 t + \vartheta_1) + C_2 \cos(\Omega_2 t + \vartheta_2) + C_3 \cos(3\Omega_1 t + \vartheta_3) + C_4 \cos(3\Omega_2 t + \vartheta_4),$$

and the solution also containing combined subharmonic components,

$$\begin{aligned} y(t) = {} & C_1 \cos(\Omega_1 t + \vartheta_1) + C_2 \cos(\Omega_2 t + \vartheta_2) + C_3 \cos(3\Omega_1 t + \vartheta_3) \\ & + C_4 \cos(3\Omega_2 t + \vartheta_4) + b^{(1)} \cos(\tfrac{1}{2}(\Omega_1 + \Omega_2)t + \vartheta^{(1)}) + b^{(2)} \cos(\tfrac{1}{2}(\Omega_1 - \Omega_2)t + \vartheta^{(2)}) \\ & + b^{(3)} \cos(\tfrac{1}{2}(\Omega_1 + 3\Omega_2)_t + \vartheta^{(3)}) + b^{(4)} \cos(\tfrac{1}{2}(\Omega_1 - 3\Omega_2)t + \vartheta^{(4)}). \end{aligned}$$

For the chaotic behaviour the amplitude probability density functions are "chaotic" with multiple maxima, as there exist values of the amplitude which are more probable and occur more often in the time history than neighbouring values. As the behaviour is not regular, the probability of each value of the amplitude is not constant and depends on time: see Figure 8. This causes the shapes of the probability density functions to depend on the period of time for which the time history is considered, or for which the Fokker–Planck–Kolmogorov equation is solved.

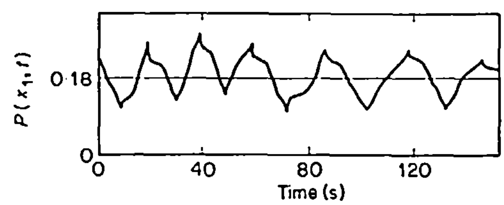

Figure 8. $P(x_1, t)$ versus time for constant amplitude $x_1 = 0{\cdot}5$.

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