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CHAOS IN A NOISY MECHANICAL SYSTEM WITH STRESS RELAXATION

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The influence of periodic and random external excitation on the chaotic behaviour of a self-excited system is considered. The Lyapunov exponents of the noisy system have been defined as random variables. The properties of their distribution which allow one to quantify chaos have been determined.

1. INTRODUCTION

Chaotic behaviour is known to occur in a variety of relatively simple deterministic mechanical systems [1-9]. In the present paper the influence of the random external excitation on the chaotic behaviour of the mechanical system with stress relaxation shown in Figure 1 is investigated.

The equations of motion of the mass m are

$$m\ddot{x}+\delta=F(t,\omega),$$
$$\dot{\delta}+a\delta=bx+c\dot{x}x^2+ex+dx^2+fx^3, \tag{1}$$

where $a=1/\bar{a}$, $\omega\in\Omega$ and (Ω,β,μ) is a probabilistic space in which Ω, β and μ are respectively, the set of random variables, the σ-field of its Borel subsets and the probabilistic measure.

In the absence of the external excitation ($F(t,\omega)=0$), self-excited oscillations of the system (1) occur. By calculating the maximum one-dimensional Lyapunov exponent [10-14] it has been found that the chaotic behaviour occurs when the parameters a, c and d are those corresponding to the zone indicated in Figure 2 and the parameters b, e and f satisfy the following relations: $b=\alpha-a$, $f=c/3+\xi d$ and $e=\alpha-\gamma d$ where $\alpha\in(2{\cdot}2,3{\cdot}9)$, $\xi\in(0{\cdot}65,0{\cdot}9)$ and $\gamma\in(0{\cdot}9,1{\cdot}2)$. Without loss of generality in the calculations described here $m=1$ was taken.

Consider now the influence of a random external excitation on the chaotic behaviour of the self-excited system. The external excitation $F(t,\omega)$ is assumed to have the form

$$F(t,\omega)=A\cos\Omega_0 t+\eta(t,\omega), \tag{2}$$

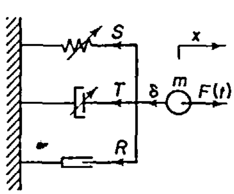

Figure 1. Model of the system: m, mass; x, displacement of the mass m; δ, internal stress; S, resistance due to the stiffness, $S=(ex+dx^2+fx^3)\bar{a}$; T, resistance due to the damping, $T=(bx+cxx^2)\bar{a}$; R, resistance due to stress relaxation, $R=-\bar{a}\dot{\delta}$; $F(t,\omega)$, external force.

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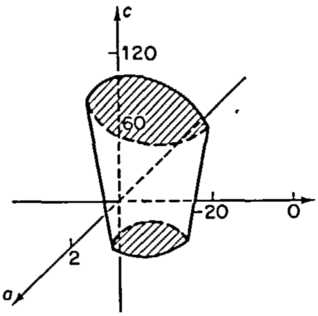

Figure 2. Chaotic zone in the parameters space.

where A and Ω_0 are constant and $\eta(t, \omega)$ is a "band-limited white noise" stochastic process with zero mean and spectral density:

$$S_0(\nu)=\begin{cases}\dfrac{\sigma^2}{\nu_{max}-\nu_{min}} & \text{for} \quad \nu\in[\nu_{min}, \nu_{max}] \\ 0 & \text{for} \quad \nu\notin[\nu_{min}, \nu_{max}]\end{cases}. \tag{3}$$

σ is the variance of the process $\eta(t, \omega)$ and $[\nu_{min}, \nu_{max}]$ is the interval of the frequencies considered.

Recently a method in which the amplitude probability density function [15-17] has been developed to investigate the chaotic behaviour of non-linear systems perturbed by random noise. In what follows an attempt to compute the maximum one-dimensional Lyapunov exponents for such systems is described.

2. RANDOM LYAPUNOV EXPONENTS

External excitation (2) causes the equations of motion (1) to be non-autonomous and because of the random component $\eta(t, \omega)$ it will be impossible to obtain the variational equations directly and use standard methods of computation of Lyapunov exponents. Instead one can do this by the following approximation of the random function $\eta(t, \omega)$.

The stochastic process $\eta(t, \omega)$ can be approximated as a sum of harmonic components $\eta_R(t)$, where

$$\eta_R(t)=\sum_{k=1}^{K} A_k \cos(\nu_k t+\varphi_k). \tag{4}$$

φ_k $(k=1, 2, \ldots, K)$ are independent random variables with uniform distribution on the interval $[0, 2\pi]$. There are a few methods of calculating A_k and ν_k, such as those of Borgman [18], Rice (see reference [19]), Shinozuka [19] and Wróbel [20]. In these calculations Rice's method was used with A_k and ν_k deterministic and given by

$$A_k=\sqrt{2S_0(\nu_k)\,\Delta\nu}, \qquad \nu_k=(k-0{\cdot}5)\,\Delta\nu+\nu_{min}+\delta\nu_k, \qquad \Delta\nu=(\nu_{max}-\nu_{min})/K, \tag{5}$$

where $\delta\nu_k$ are uniformly distributed on the interval $[-0{\cdot}1\Delta\nu, 0{\cdot}1\Delta\nu]$. A unique realization of the process $\eta_R(t)$ is obtained by selecting the random variables φ_k and substituting them into equation (4). As a measure of the quality of these realizations the mean square error ε of the spectral density of the process $\eta(t, \omega)(S_0(\nu))$ and the spectral density of its realization $\eta_R(t)(S(\nu))$

$$\varepsilon=\frac{1}{\nu_{max}-\nu_{min}}\left\{\int_{\nu_{min}}^{\nu_{max}}[S(\nu)-S_0(\nu)]^2\,d\nu\right\}^{1/2} \tag{6}$$

can be taken.

In the case of external excitation (2) and approximation (4), and after the transformation $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \Omega_0 t$, $x_4 = \nu_1 t + \varphi_1, \ldots, x_{3+K} = \nu_K t + \varphi_K$, equations (1) take the form

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = (1/m)a\delta + A\cos x_3 + B[\cos x_4 + \cos x_5 + \cdots + \cos x_{K+3}],$$
$$\dot{x}_3 = \Omega_0, \qquad x_4 = \nu_1, \qquad x_5 = \nu_2, \ldots$$
$$\dot{x}_{3+K} = \nu_{3+K}, \qquad \dot{\delta} = -a\delta + bx_2 + cx_2x_1^2 + ex_1 + dx_1^2 + fx_1^3, \tag{7}$$

with initial conditions $x_{30} = 0, x_{40} = \varphi_1, x_{50} = \varphi_2, \ldots, x_{3+K0} = \varphi_K$. The variational equations are as follows:

$$\dot{y}_1 = y_2,$$
$$\dot{y}_2 = (1/m)y_\delta + A(\sin x_3)y_3 + B[(\sin x_4)y_4 + (\sin x_5)y_5 + \cdots + (\sin x_{3+K})y_{3+K}],$$
$$y_3 = 0, \qquad \dot{y}_4 = 0, \ldots, \dot{y}_{3+K} = 0,$$
$$\dot{y}_\delta = -ay_\delta + by_2 + 2cx_1x_2y_1 + cx_1^2y_2 + ey_1 + 2dx_1y_1 + 3fx_1^2y_1. \tag{8}$$

Without loss of generality one can put $y_3, y_4, \ldots, y_{3+K} = 1$ which simplifies the system (8).

The Lyapunov one-dimensional exponent for the noisy system (1) can be described as the following random variable:

$$\lambda(x_{10}, x_{20}, \delta_0, y_{10}, y_{20}, y_{\delta 0}, \omega) = \lim_{t\to\infty} \frac{1}{t} \ln \|y(t)\|. \tag{9}$$

The Lyapunov exponent is independent of the norm and the following one was used: $\|y\| = \sum_{n=1}^{2} |y_n|$. The largest rate of the Lyapunov exponent is selected by variation of the initial values $x_{10}, x_{20}, \delta_0, y_{10}, y_{20}, y_{\delta 0}$ with the same set of values φ_k.

3. INFLUENCE OF THE EXTERNAL EXCITATION ON THE CHAOS IN THE SELF-EXCITED SYSTEM

First, the system with only periodic deterministic excitation was investigated. For the parameter values for which an autonomous system ($F(t, \omega) = 0$) shows chaotic behaviour and for the particular values of A and Ω_0 the zone where the system is not chaotic was found; see Figure 3. In zone A the solution is periodic with a period $4\pi/\Omega_0$ and in zone B it is periodic with a period $2\pi/\Omega_0$.

In the zones A and B there are subzones A1 and B1 where the phase trajectories are double-revolving (see Figure 4) and subzones A2 and B2 with triple-revolving phase trajectories (see Figure 5). An example of the chaotic phase trajectory is shown in Figure 6.

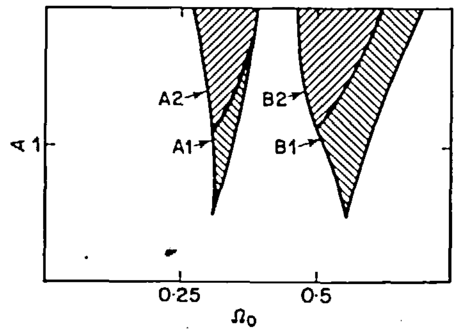

Figure 3. The influence of the periodic deterministic excitation on the chaotic behaviour: $a = 1{\cdot}75$, $b = 1{\cdot}9$, $c = 54{\cdot}2$, $d = -20{\cdot}0$, $e = 21{\cdot}2$, $f = 2{\cdot}1$.

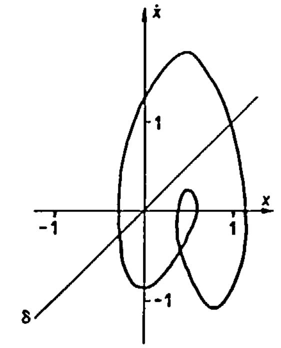

Figure 4. Double-revolving phase trajectory: $A=1{\cdot}5$, $\Omega_0=0{\cdot}6$.

In the case of random excitation different values of the Lyapunov exponents are obtained depending on the realization of the random process $\eta(t, \omega)$. Using these values one cannot determine chaotic or regular behaviour of the system. Taking into account the probability density function of the Lyapunov exponents one can obtain two types of distribution, as shown in Figure 7.

They were obtained from 100 realizations of the process $\eta(t, \omega)$. Each realization consists of $K=30$ harmonic components, and for all of them the mean square error of the spectral densities ε is less than $0{\cdot}1$.

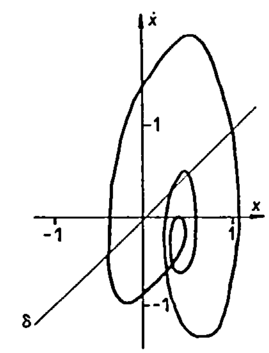

Figure 5. Triple-revolving phase trajectory: $A=1{\cdot}5$, $\Omega_0=0{\cdot}5$.

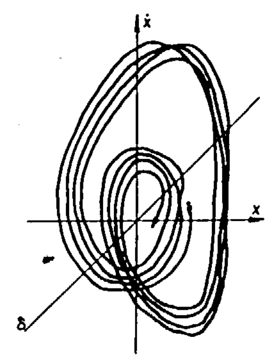

Figure 6. Chaotic phase trajectory: $A=1{\cdot}5$, $\Omega_0=0{\cdot}32$.

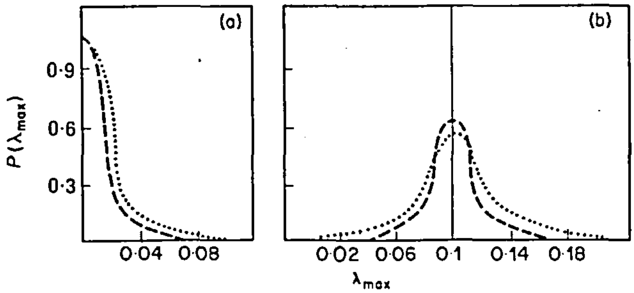

Figure 7. Distributions of random maximum one-dimensional Lyapunov exponent. (a) Regular behaviour; (b) chaotic behaviour. - - -, $\sigma=0{\cdot}05$; . . . . . , $\sigma=0{\cdot}1$.

As was checked by the method involving use of the amplitude probability density function [15-17] the first type of Lyapunov exponents distribution (see Figure 7(a)) corresponds to the regular behaviour of the system and the second one (see Figure 7(b)) to the chaotic behaviour.

In Figure 8 the zones of chaotic behaviour of the system are shown for different values of noise variance and in Figure 9 for different intervals of noise frequencies. As in the deterministic case (see again Figure 3) in the zones A and B the behaviour of the system is not chaotic. It is interesting that in the random case these non-chaotic zones increase with an increase of the noise variance.

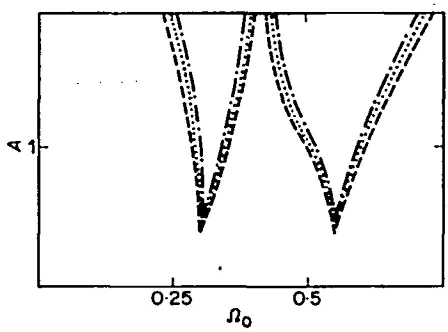

Figure 8. The influence of the noise on the chaotic behaviour of the system (1) for different values of noise variance. —·—, $\sigma=0{\cdot}05$; . . . . . , $\sigma=0{\cdot}1$; - - -, $\sigma=0{\cdot}2$.

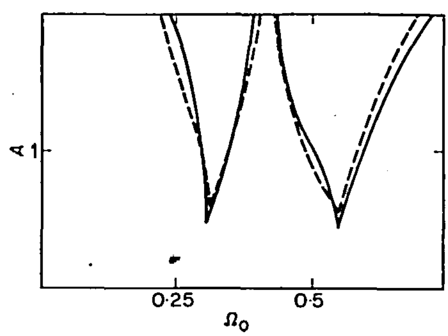

Figure 9. The influence of the noise on the chaotic behaviour of the system (1) for different values of the frequency interval. ——, $\nu_{min}=0{\cdot}4$, $\nu_{max}=1{\cdot}8$; - - -, $\nu_{min}=0{\cdot}8$, $\nu_{max}=1{\cdot}4$.

4. CONCLUSIONS

Based on the properties of the maximum one-dimensional Lyapunov exponents estimates have been made of the parameter values for which an autonomous self-excited system with stress relaxation shows chaotic behaviour.

The investigation of the influence of the periodic deterministic excitation on the chaotic behaviour of the self-excited system shows that there exist values of A and Ω_0 for which the solution is periodic with the period or double period of the external excitation.

Finally, a method has been developed of calculating Lyapunov exponents for the system with noise. Application of this method led to the finding that in the presence of external noise the chaotic zone of the system decreases.

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