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LETTERS TO THE EDITOR

RESPONSE OF A NON-LINEAR SYSTEM TO TWO-STEP MARKOV NOISE: NON-MARKOVIAN AVERAGING

1. INTRODUCTION

The study of the influence of random forces on non-linear systems is a very practical problem. Problems of this type are of great importance in many branches of science and engineering because no real physical system is free of noise and most of them are non-linear. The increased interest in stochastic systems has stimulated not only mathematical works [1, 2] but also investigations of applications [3–5].

In recent years the stochastic averaging method has been developed. This method is based on the well-known Krylov–Bogoliubov–Mitropolskii technique for solving deterministic non-linear vibration problems [6] and the Stratonovich–Khasiminskii limit theorem for stochastic differential equations [4]. This method allows one to calculate the Markovian approximate response of the system and requires the fulfilment of the following assumptions. (1) The damping is light and the envelope of the excitation power spectrum is scaled accordingly, so the oscillation varies slowly with respect to time and can be treated as a constant over an appropriate period of oscillation. Consequently, oscillatory terms can be approximated by their temporal averages over one period of oscillation. (2) With light damping and a broadband random excitation the relaxation time of the oscillator response is much greater than the correlation time of the excitation, so it is possible to model the power input due to the excitation as a non-zero mean component plus an additional fluctuating component with the character of white noise. The first assumption is the same as in deterministic procedure; the second one is necessary to satisfy the conditions of the Stratonovich–Khasiminskii limit theorem.

In what follows an averaging procedure which differs from Markovian one in the second of the above assumptions is described. In a non-Markovian case the fluctuating component has the form of a non-Gaussian two-step Markov noise which can be useful for many problems of automatic control. An example of its realization is shown in Figure 1. It has a zero-mean and the correlation function is

$$\langle \eta(t)\eta(t')\rangle = \Delta^2 \exp[-\lambda|t-t'|], \qquad (1)$$

where Δ and λ are constant, $\langle\cdot\rangle$ indicates an ensemble average, and there are two possible values $\pm\Delta$ with equal probability, and jumps with probability $\frac{1}{2}\lambda\, dt$ for each dt.

In the case of this form of fluctuation the system response is non-Markovian and an approximate solution can be obtained based on the theory of non-Markovian stochastic

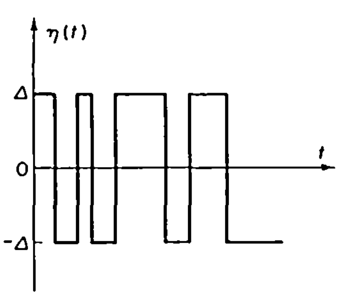

Figure 1. Realization of the two-step Markov noise.

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differential equations [7, 8] in a similar way as in references [9, 10]. The closed system of linear partial differential equation for the probability density of the response can be obtained. In the case of a stationary state this system has a solution given in explicit form which can be used for many practical problems. This method allows one to determine the dependence of the response on both parameters of the stochastic process: Δ and λ.

In the white noise limit of the process $\eta(t)$ the non-Markovian averaging procedure gives the same results as ordinary stochastic averaging.

2. AVERAGING PROCEDURE

Consider an autonomous oscillating system with one degree of freedom which is subject to two-step Markov noise $\eta(t)$ and is described by the second order differential equation

$$d^2x/dt^2+\omega^2 x=\varepsilon f_1(x, dx/dt)+\sqrt{\varepsilon}\sigma f_2(x, dx/dt)\eta(t), \tag{2}$$

where ε is a small parameter, ω and σ are constants, and f_1 and f_2 are non-linear functions satisfying all the necessary conditions, which can depend periodically upon the time t. Equation (2) is a quasilinear differential stochastic equation of a two dimensional non-Markovian process.

Due to the smallness of the parameter ε, the application of Krylov-Bogoliubov-Mitropolskii asymptotic method is possible. This method leads from equation (2) to a system of first order equations in terms of the amplitude and phase of random oscillations. To obtain this system the following change of variables is done:

$$x=a(t)\cos\psi, \qquad dx/dt=-a(t)\omega\sin\psi, \qquad \psi=\omega t+\varphi(t). \tag{3}$$

Here the amplitude envelope process $a(t)$ and phase process $\varphi(t)$ are slowly varying with respect to time, when ε is small. In terms of a and φ equation (2) can be cast into the following pair of exact equations:

$$\begin{aligned} da/dt &= -(\varepsilon/\omega)f_1(a\cos\psi, -a\omega\sin\psi)\sin\psi \\ &\quad -(\sqrt{\varepsilon}\sigma/\omega)f_2(a\cos\psi, -a\omega\sin\psi)\sin\psi\,\eta(t), \\ d\varphi/dt &= -(\varepsilon/a\omega)f_1(a\cos\psi, -a\omega\sin\psi)\cos\psi \\ &\quad -(\sqrt{\varepsilon}\sigma/a\omega)f_2(a\cos\psi, -a\omega\sin\psi)\cos\psi\,\eta(t). \end{aligned} \tag{4}$$

Equations (4) are equivalent to equations (2) and also represent a two-dimensional non-Markovian process. The complete averaging according to Kolomietz [11] can be applied and as the averaged system of stochastic differential equations, corresponding to system (4), one will obtain

$$da/dt=\bar{f}_1(a)+\bar{f}_2(a)\eta(t), \qquad d\varphi/dt=\bar{f}_3(a)+\bar{f}_4(a)\eta(t), \tag{5, 6}$$

where

$$\bar{f}_1(a)=-\int_0^{2\pi}\frac{\varepsilon}{\omega}f_1(a\cos\psi, -a\omega\sin\psi)\sin\psi\,d\psi,$$

$$\bar{f}_2(a)=\sqrt{\int_0^{2\pi}\frac{\varepsilon\sigma^2}{\omega^2}f_2^2(a\cos\psi, -a\omega\sin\psi)\sin^2\psi\,d\psi},$$

$$\bar{f}_3(a)=-\int_0^{2\pi}\frac{\varepsilon}{a\omega}f_1(a\cos\psi, -a\omega\sin\psi)\cos\psi\,d\psi,$$

$$\bar{f}_4(a)=\sqrt{\int_0^{2\pi}\frac{\varepsilon\sigma^2}{a^2\omega^2}f_2^2(a\cos\psi, -a\omega\sin\psi)\cos^2\psi\,d\psi}.$$

Under certain conditions the solution of system (5) is reduced to that of system (4) in the sense of root-mean-square convergence of stochastic variables.

3. PROBABILITY DENSITY OF THE RESPONSE

An inspection of equation (5) shows that the amplitude process a is uncoupled from the phase process φ: i.e., a is a one-dimensional non-Markovian process. The probability density function for a, $P(a, t\,|\,a_0, t_0)$ where a_0 and t_0 are initial conditions, can be obtained from the stochastic Liouville equation [1] for the density $p(a, t\,|\,a_0, t_0)$ of a set of realizations of equation (5):

$$\dot{p}(a, t\,|\,a_0, t_0) = -(\partial/\partial a)\{\bar{f}_1(a) + \bar{f}_2(a)\eta(t)\}p(a, t\,|\,a_0, t_0). \tag{7}$$

Taking the average over $\eta(t)$ and using Van Kampen's lemma [8]

$$P(a, t\,|\,a_0, t_0) = \langle p(a, t\,|\,a_0, t_0)\rangle, \tag{8}$$

one obtains

$$\partial P(a, t\,|\,a_0, t_0)/\partial t = -(\partial/\partial a)\bar{f}_1(a)P(a, t\,|\,a_0, t_0) - (\partial/\partial a)\bar{f}_2(a)P_1(a, t\,|\,a_0, t_0), \tag{9}$$

where $P_1(a, t\,|\,a_0, t_0) = \eta(t)p(a, t\,|\,a_0, t_0)$. Since $p(a, t\,|\,a_0, t_0)$ is a functional of $\eta(t)$, one can use the formula of differentiation of Shapiro and Loginov [1],

$$(\partial/\partial t)\langle\eta(t)\Phi[\eta(t)]\rangle = -\lambda\langle\eta(t)\Phi[\eta(t)]\rangle + \langle\eta(t)(\partial/\partial t)\Phi[\eta(t)]\rangle, \tag{10}$$

where $\Phi[\eta(t)]$ is a functional of $\eta(t)$ and the average is over the distribution of $\eta(t)$, to obtain an equation for $P_1(a, t\,|\,a_0, t_0)$:

$$\frac{\partial P_1(a, t\,|\,a_0, t_0)}{\partial t} = -\lambda P_1(a, t\,|\,a_0, t_0) - \frac{\partial}{\partial a}\bar{f}_1(a)P_1(a, t\,|\,a_0, t_0) - \frac{\partial}{\partial a}\bar{f}_2(a)\Delta^2 P(a, t\,|\,a_0, t_0). \tag{11}$$

In equation (11) the fact has been used that the square of $\eta(t)$ is constant: $\eta^2(t) = \Delta^2$.

Equations (9) and (11) give a closed system of linear partial differential equations, the solution of which will give the amplitude probability density function $P(a, t\,|\,a_0, t_0)$ provided that the initial condition $P(a, t\,|\,a_0, t_0)|_{t=t_0}$ is known, and also another initial condition because there are two linear equations. The second condition obtained by assuming statistical independence between $\eta(t)$ and $p(a, t\,|\,a_0, t_0)$ at $t = t_0$; i.e.,

$$\langle\eta(t)p(a, t\,|\,a_0, t_0)\rangle|_{t=t_0} = P_1(a, t_0\,|\,a_0, t_0) = 0 \tag{12}$$

which implies in equation (9) that

$$\partial P(a, t\,|\,a_0, t_0)/\partial t) - (\partial/\partial a)\bar{f}_1(a)P(a, t\,|\,a_0, t_0)|_{t=t_0} = 0. \tag{13}$$

This, together with,

$$P(a, t\,|\,a_0, t_0)|_{t=t_0} = \delta(a - a_0) \tag{14}$$

are then the initial conditions of the system (9) and (11).

Formal integration of equation (9) gives

$$P_1(a, t\,|\,a_0, t_0) = -\Delta^2\int_{t_0}^{t}\exp\left[-\left(\lambda + \frac{\partial}{\partial a}\bar{f}_1(a)\right)(t - t') - \frac{\partial}{\partial a}\bar{f}_2(a)P(a, t'\,|\,a_0, t_0)\right]\mathrm{d}t' = -\Delta^2 T(a, t), \tag{15}$$

where expression (12) has been used. Substituting expression (15) into equation (11) gives a formal equation for $P(a, t\,|\,a_0, t_0)$:

$$\frac{\partial P(a, t\,|\,a_0, t_0)}{\partial t} = -\frac{\partial}{\partial a}\bar{f}_1(a)P(a, t\,|\,a_0, t_0) + \Delta^2\frac{\partial}{\partial a}\bar{f}_2(a)T(a, t). \tag{16}$$

The time-dependent solution is not always known, but one can obtain the stationary state solution in the form:

$$P_{st}(a\,|\,a_0) = N\frac{\bar{f}_2(a)}{\Delta^2\bar{f}_2^2(a) - \bar{f}_1^2(a)}\exp\left\{\lambda\int_{a_0}^{a}\frac{\bar{f}_1(a')\,\mathrm{d}a'}{\Delta^2\bar{f}_2^2(a') - \bar{f}_1^2(a')}\right\}, \tag{17}$$

where the constant N is found by using the normalization function $P_{st}(a\,|\,a_0)$.

If, in equation (15), one puts

$$\exp[-\lambda\,|t - t'|] \simeq (2/\lambda)\,\delta(t - t') \tag{18}$$

one obtains the white noise limit for $a(t)$. This limit holds for $\lambda \to \infty$, $\Delta \to \infty$ and Δ^2/λ finite. In this case equation (16) has the standard form of the Fokker-Planck-Kolmogorov equation:

$$\frac{\partial P(a, t\,|\,a_0, t_0)}{\partial t} = -\frac{\partial}{\partial a}\bar{f}_1(a)P(a, t\,|\,a_0, t_0) + \tfrac{1}{2}\Delta^2\frac{\partial^2}{\partial a^2}\bar{f}_2^2(a)P(a, t\,|\,a_0, t_0). \tag{19}$$

In the white noise limit of a Gaussian process $\eta(t)$ the non-Markovian averaging method converges to the Markovian one.

By following the same method (equations (7)-(11)) it is possible to obtain the equations for the joint probability density of the amplitude a and the phase φ. In this case the equations equivalent to equations (9) and (11) have the form

$$\begin{aligned}\frac{\partial P(a, \varphi, t\,|\,a_0, \varphi_0, t_0)}{\partial t} = &-\frac{\partial}{\partial a}\bar{f}_1(a)P(a, \varphi, t\,|\,a_0, \varphi_0, t_0)\\ &-\frac{\partial}{\partial \varphi}\bar{f}_3(a)P_1(a, \varphi, t\,|\,a_0, \varphi_0, t_0)\\ &+\frac{\partial}{\partial a}\bar{f}_2(a)P(a, \varphi, t\,|\,a_0, \varphi_0, t_0)\\ &+\frac{\partial}{\partial \varphi}\bar{f}_4(a)P_1(a, \varphi, t\,|\,a_0, \varphi_0, t_0),\end{aligned} \tag{20}$$

where $P_1(a, \varphi, t\,|\,a_0, \varphi_0, t_0) = \langle\eta(t)p(a, \varphi, t\,|\,a_0, \varphi_0, t_0)\rangle$ and

$$\begin{aligned}&\partial P_1(a, \varphi, t\,|\,a_0, \varphi_0, t_0)/\partial t\\ &\quad= -\lambda P_1(a, \varphi, t\,|\,a_0, \varphi_0, t_0) - (\partial/\partial a)\bar{f}_1(a)P_1(a, \varphi, t\,|\,a_0, \varphi_0, t_0)\\ &\quad\quad-(\partial/\partial\varphi)\bar{f}_3(a)P_1(a, \varphi, t\,|\,a_0, \varphi_0, t_0) - (\partial/\partial a)\bar{f}_2(a)\Delta^2P(a, \varphi, t\,|\,a_0, \varphi_0, t_0)\\ &\quad\quad-(\partial/\partial\varphi)\bar{f}_4(a)\Delta^2P(a, \varphi, t\,|\,a_0, \varphi_0, t_0).\end{aligned} \tag{21}$$

The initial conditions are equivalent to expressions (12) and (14):

$$P_1(a, \varphi, t\,|\,a_0, \varphi_0, t_0)|_{t=t_0} = 0, \qquad P(a, \varphi, t\,|\,a_0, \varphi_0, t_0)|_{t=t_0} = \delta(a - a_0)\,\delta(\varphi - \varphi_0). \tag{23, 23}$$

In general, the analytical solution of equations (20) and (21) is unknown but these equations can be very useful in numerical analysis of many vibration problems.

4. EXAMPLE

As an example consider Van der Pol's equation subjected to the two-step Markov noise in the form

$$\mathrm{d}^2t/\mathrm{d}t^2+(1+\sqrt{\varepsilon}\sigma\eta(t))x=\varepsilon(1-x^2)\,\mathrm{d}x/\mathrm{d}t. \tag{24}$$

Application of the averaging procedure gives the following averaged equation for the amplitude envelope process:

$$\mathrm{d}a/\mathrm{d}t=(\varepsilon a/2)(1-a^2/4)+(\sqrt{\varepsilon}\sigma a/2\sqrt{2})\eta(t). \tag{25}$$

This has the same form as equation (5), so by taking into account the initial conditions (12)–(14) the amplitude probability density function for a stationary state can be calculated from equation (17).

Numerical examples of the amplitude probability function for different values of λ^{-1} are shown in Figure 2. It is found that with the increase of the correlation time the mean value of the amplitude increases.

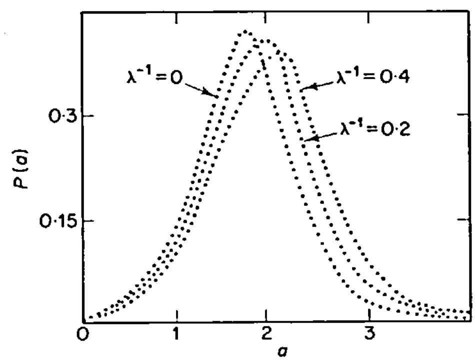

Figure 2. Amplitude probability density function for different values of λ^{-1}: $\sigma=1$, $\varepsilon=0{\cdot}1$, $\Delta=0{\cdot}1$.

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