# Chaos in a limit-cycle system with almost periodic excitation



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## LETTER TO THE EDITOR

# Chaos in a limit-cycle system with almost periodic excitation

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**Abstract.** The influence of the almost periodic excitation on the chaotic behaviour of Van der Pol's oscillator is reported.

Recently systems with chaotic behaviour have attracted increasing attention [1-10]. Several examples of chaotic solutions have also been found for limit-cycle systems, which appear in the mathematical analysis of many phenomena (lasers, the biochemical oscillator, many engineering problems and in particular the mechanical system with dry friction [8-10]). The best known example is Van der Pol's oscillator

$$\ddot{x}+d(x^2-1)\dot{x}+x^n=0 \tag{1}$$

which serves as a basic model of self-excited oscillations.

In the present letter we study the above limit-cycle system under the influence of an almost periodic external excitation

$$a\cos(\omega t)\cos(\Omega t) \tag{2}$$

where $a, \Omega$ and $\omega$ are constant, $\Omega$ and $\omega$ being incommensurate.

In the case of $\omega=0$ we obtain the typical system with periodic excitation, which was investigated by various authors [8-10]. Particularly for $n=1$ the chaotic behaviour has been found by Parlitz and Lauterborn [10] for the following values of system parameters:

$$a=d=5.0 \qquad \omega\in[2.463, 2.466].$$

We try to answer the question: what happens when the excitation is almost periodic? Some experiments with almost periodic excitation have been already done. For example in [11] the authors consider the transmission to chaos in an electronic Josephson-junction simulator driven by two independent AC sources. The influence of the two external periodic excitations on the chaotic behaviour of an anharmonic oscillator has been investigated in [12-14] and on the non-linear pendulum in [15].

For characterising the chaotic behaviour we consider Lorenz plots (recursive plots of $x$ in the $(x[n], x[n+1])$ plane, where $x[n]$ $(n=1,2,\ldots)$ is the extremum of the oscillation waveform), power spectra and maximum Lyapunov exponents. We consider $\Omega=2.464$ and $\omega\in[0, 2.464]$. All the numerical simulations are done using the modified Runge-Kutta method of the fourth order. The calculation step is $\pi/100\Omega$. The FFT procedure is used for power spectra.

The evolution of the strange attractor depending on $\omega$ is described by means of the Lorenz plots and power spectra of figure 1.

In figure 1($a$) we have the strange attractor for $\omega=0$. Chaotic behaviour is observed in figures 1($b$, $c$) up to $\omega=1.0$ but the structure of the attractor becomes simpler with

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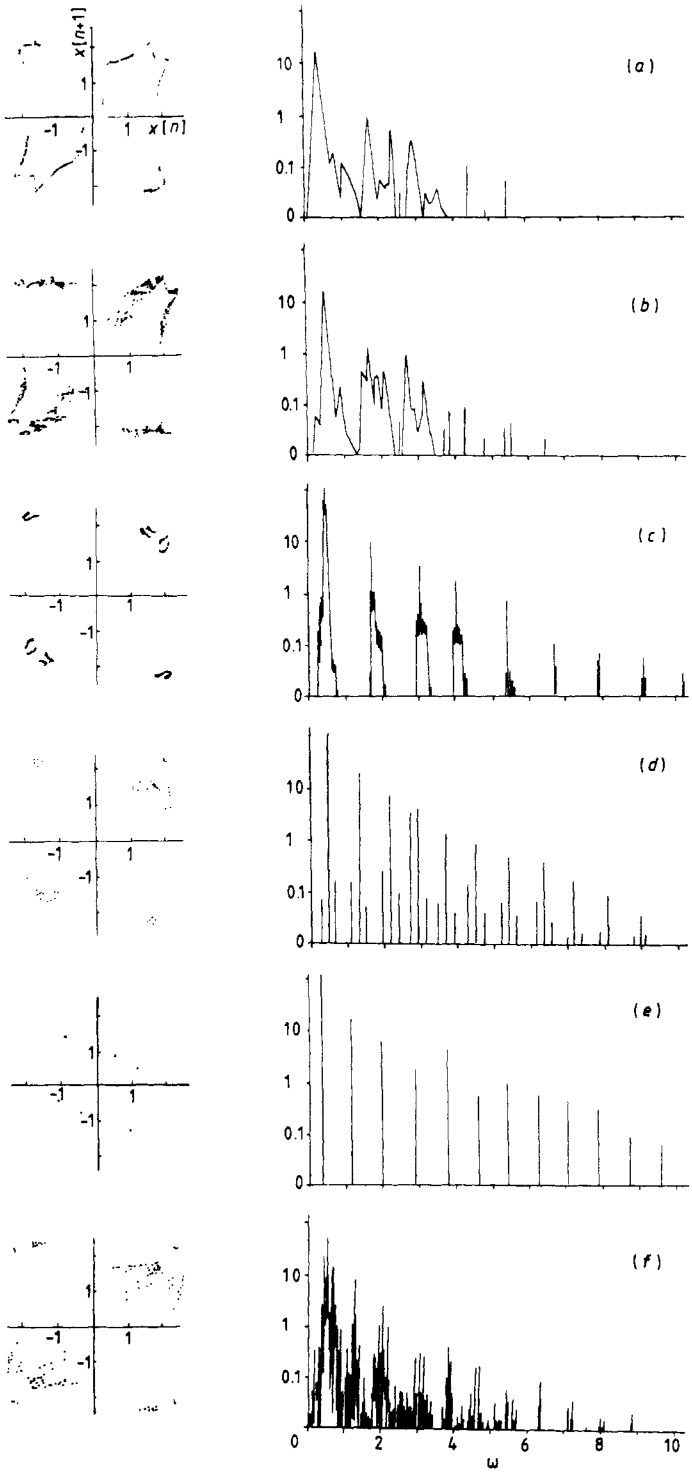


**Figure 1.** Lorenz plots of $x[n+1]$ against $x[n]$ and power spectra of the system (1) with excitation (2) with parameters $a = d = 5.0$; $\Omega = 2.464$ for ($a$) $\omega = 0.0$, ($b$) 0.244, ($c$) 0.601, ($d$) 1.173, ($e$) 1.232, ($f$) 1.236, ($g$) 1.4, ($h$) 1.6, ($i$) 1.8, ($j$) 1.848, ($k$) 1.9, ($l$) 2.4.

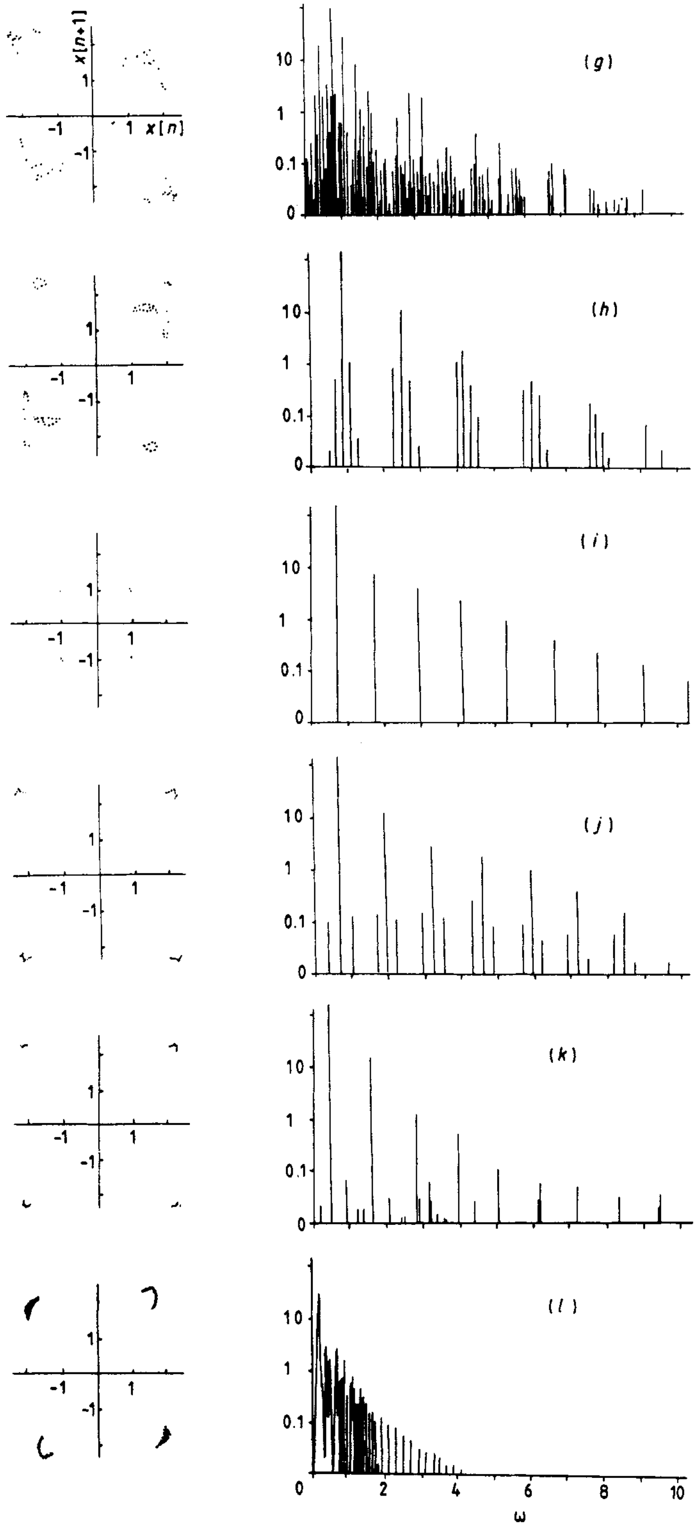


**Figure 1.** (continued)

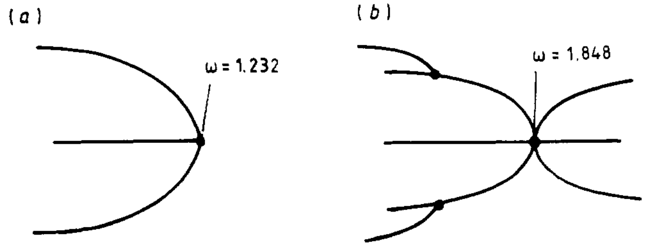


**Figure 2.** Diagram of frequency behaviour in the neighbourhood of (*a*) $\omega/\Omega = \frac{1}{2}$ and (*b*) $\omega/\Omega = \frac{3}{4}$.

**Table 1.** Positive maximum Lyapunov exponent for different values of $\omega$.

| $\omega$ | $\alpha_{max}$ |
|---|---|
| 0.0 | 0.22 |
| 0.244 | 0.13 |
| 0.601 | 0.03 |
| 1.26 | 0.08 |
| 2.4 | 0.06 |

increasing $\omega$. One of the simplest attractor structures is observed for $\omega = 1.2315$ where the Lorenz plot consists of six points and fourteen frequencies are visible in the power spectra (see figure 1(*e*)). In the close neighbourhood of $\omega = 1.232$ for lower values of $\omega$ we observe trifurcation of each of the frequencies in figure 1(*e*) (compare figures 1(*e*) and 1(*d*)) as diagrammatically shown in figure 2(*a*). This trifurcation is not visible for higher values of $\omega$ where, for example for $\omega = 1.236$, the behaviour of the system is again chaotic (see figure 1(*f*)). The chaotic behaviour is lost for the second time for $\omega = 1.4$. For this value of $\omega$ we have almost periodic behaviour of the very complicated form (the so-called non-chaotic strange attractor [15]) shown in figure 1(*g*). With further increase of the value of $\omega$ the structure of the attractor becomes simpler as in figures 1(*h*–*j*). The mechanism of this simplification for the main frequencies is shown in figure 2(*b*). The same trifurcation is also visible for $\omega > 1.847$ (compare figures 1(*j*) and 1(*k*)). For larger values of $\omega$ the attractor becomes more complicated and chaotic behaviour is again observed in the neighbourhood of $\omega = 2.4$ in figure 1(*l*). The values of positive maximum Lyapunov exponents for the $\omega$ values of figure 1 are presented in table 1. To summarise the results presented above we found that the chaotic behaviour is weakened by the existence of the second frequency in the formula describing the excitation force. This weakening for Van der Pol's equation is even stronger than in the case of Duffing's equation [13]. It is interesting that the simplest shape of attractor is obtained for the same ratio of $\omega/\Omega = \frac{1}{2}$ and $\omega/\Omega = \frac{3}{4}$, for which Duffing oscillators, excited in the same form, lose chaotic properties [13]. We hope to explain this problem in future work.

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