Journal of Sound and Vibration (1990) **138**(1), 162–169

STRANGE NON-CHAOTIC ATTRACTORS OF A QUASI-PERIODICALLY FORCED VAN DER POL'S OSCILLATOR

1. INTRODUCTION

Theoretical investigations of periodically forced non-linear systems have been of great interest from a number points of view for many years. Recently some investigations of systems with quasi-periodic forcing has appeared: Steeb *et al.* 1986 [1], Kapitaniak *et al.* 1987 [2], Romeiras and Ott 1988 [3], Kapitaniak 1988 [4], Kapitaniak and Wojewoda 1988 [5], Wiggins 1987 [6]. In these investigations, besides the typical behaviour of periodically forced systems, some new phenomena were found such, as weakening of chaos [1, 2], combined bifurcations [4], and strange non-chaotic attractors [3, 5], which seem to be characteristic for this kind of system.

In what follows, we give new examples of strange non-chaotic attractors of the Van der Pol oscillator

$$\ddot{x}+d(\dot{x}^2-1)\dot{x}+x=a\cos(\omega t)\cos(\Omega t), \tag{1}$$

where a, d, ω and Ω are constants and describe its main properties. Equation (1) has a four-dimensional phase space: $(\dot{x}, x, \Theta_1=\omega t, \Theta_2=\Omega t)\in\mathbb{R}^2\times\mathbb{S}^1\times\mathbb{S}^1$. One can reduce the study of (1) to the study of associated three-dimensional Poincaré map obtained by defining a three-dimensional cross-section of the four-dimensional phase space by fixing the phase of one of the angular variables and allowing the remaining three variables that start on the cross-section to evolve in time under the effect of the flow generated by equation (1) until they return to the cross-section. If one fixes the phase Θ_2 the Poincaré map is defined as a set

$$M(t_0)=\{x(t_n), \dot{x}(t_n), \Theta_1(t_n)\,|\, t_n=(2\pi n/\Omega)+t_0, \quad n=1,2,\ldots\},$$

where t is initial time. To describe the surface of the Poincaré map one plots $x(t_n)$ versus $\dot{x}(t_n)$ and $x(t_n)$ versus $\Theta_1(t_n)$ mod 2π. An alternative surface can be obtained by plotting $\dot{x}(t_n)$ versus $\Theta_1(t_n)$ mod 2π.

Of course, to characterize the attractor one also uses maximum Lyapunov exponents given by

$$\lambda=\lim_{t\to\infty}\left\{\frac{1}{t}\ln\left(\frac{d(t)}{d(t_0)}\right)\right\}.$$

where $d=|y^2+\dot{y}^2]^{1/2}$ and y denotes the solution of the equation variational to equation (1).

The winding number for the orbit $x(t)$ of equation (1) defined by the limit

$$w=\lim_{t\to\infty}\left\{\frac{\alpha(t)-\alpha(t_0)}{t}\right\},$$

where $(x,\dot{x})=(r\sin\alpha, r\cos\alpha)$, is another quantity.

The frequency spectrum has been obtained by Fast Fourier Transform calculation.

2. STRANGE NON-CHAOTIC ATTRACTORS: DEFINITION

First consider the dynamics of the systems $\dot{\bar{x}}=\bar{f}(\bar{x}, t)$, where $\bar{x}=[x_1,\ldots,x_n]^{\mathrm{T}}$ represent the state variables of the n-dimensional phase space, and $\bar{f}=[f_1,\ldots,f_n]^{\mathrm{T}}$ gives the

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coupling between variables. This system is described by n Lyapunov exponents λ_i ($i = 1, 2, \ldots n$), and if $\sum_{i=1}^{n} \lambda_i \leq 0$ then the evolution of the system takes place in a limited subspace of the phase space. The attractor of the system is a specific subspace which is asympotically reached in time (or course, if $\sum_{i=1}^{n} \lambda_i > 0$ the system may never reach any attractor).

The word "strange" refers to the geometrical structure of the attractor, and an attractor which is not a finite set of points, a limit cycle (a closed curve), a smooth (piecewise smooth) surface (for example a torus), or is bounded by a piecewise smooth closed surface volume is called a strange attractor. An attractor is chaotic if at least one Lyapunov exponent is positive (typical nearby orbits diverge exponentially in time). From what was said above, one finds that a strange non-chaotic attractor is an attractor which is geometrically strange, but for which typical orbits have non-positive Lyapunov exponents.

3. MAIN PROPERTIES OF STRANGE NON-CHAOTIC ATTRACTORS

Some examples of the Poincaré surfaces are shown in Figure 1. In Figure 1(a) we have the example of periodic behaviour for $\omega = 1{\cdot}848$, Figure 1(b) represents two-frequency quasi-periodic behaviour for $\omega = 1{\cdot}614$. For the periodic and two-frequency quasi-periodic

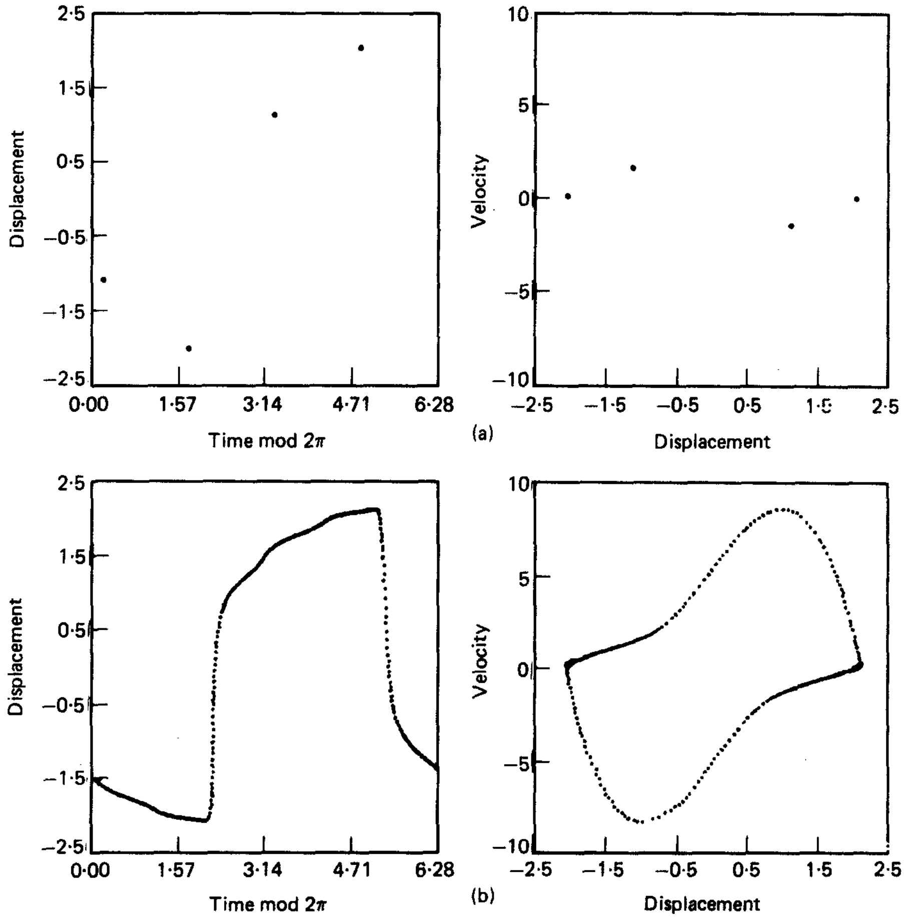

Figure 1. Poincaré surfaces of the system (1): (a) periodic attractor $\omega = 1{\cdot}848$; (b) two-frequency quasi-periodic attractor $\omega = 1{\cdot}614$; (c) strange non-chaotic attractor $\omega = 1{\cdot}401$; (d) chaotic attractor $\omega = 1{\cdot}395$.

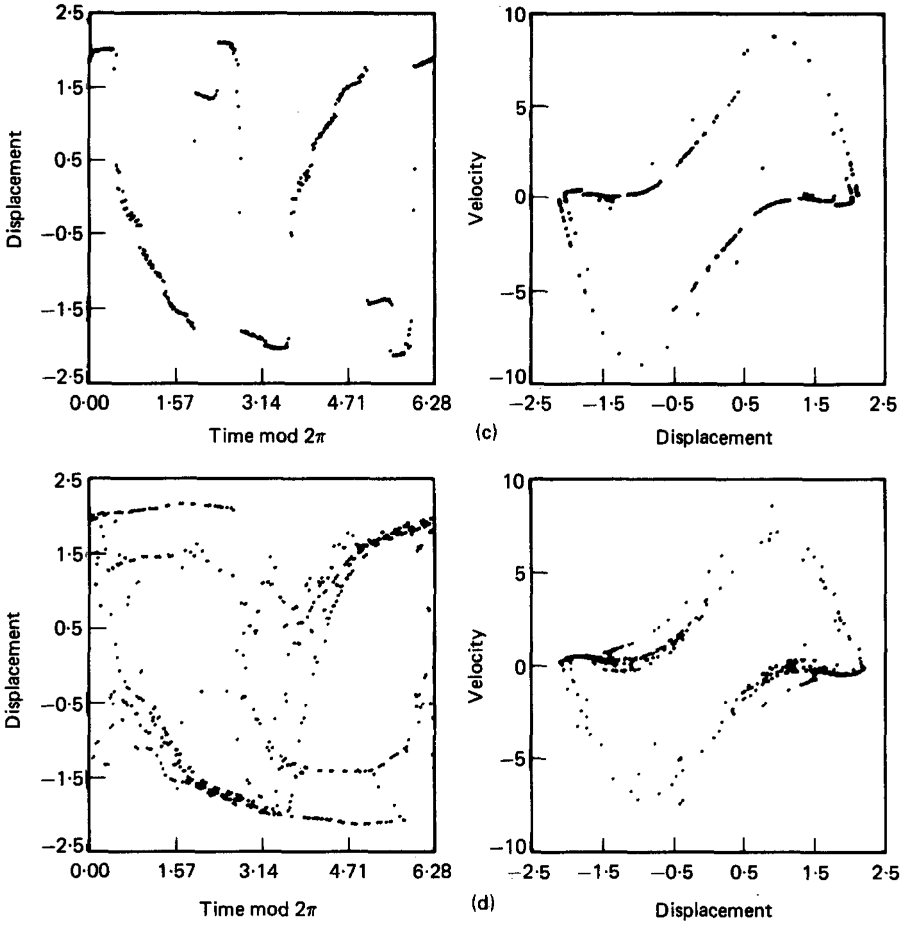

Figure 1—*continued*

behaviour we have a negative Lyapunov exponent and a winding number satisfying the relation where

$$w = (l/n)\omega + (m/n)\Omega,$$

where l, m, n are integer (in the case of periodic behaviour ω and Ω are commensurate). With further decrease of the ω the Lyapunov exponent is still negative but the winding number does not satisfy the relation (2), and we have the example of a strange non-chaotic attractor (Figure 1(c), $\omega = 1{\cdot}401$). This type of attractor occurs in a small neighbourhood of $\omega = 1{\cdot}4$ (see Figure 4 to follow). In Figure 1(d) chaotic behaviour for $\omega = 1{\cdot}395$ is shown. In all examples Ω has been 2·464.

When the winding number does not satisfy the relation (2) and the Lyapunov exponent is zero, the three frequency quasi-periodic behaviour can occur, but we have not observed it in our example.

Strange non-chaotic attractors can be quantified on the basis of frequency spectrum. In Figure 2(a)–(d) we have plotted the frequency spectra of the orbits which correspond to the Poincaré surfaces of Figure 1. The figures show that the spectra of the periodic and two-frequency quasi-periodic attractors are concentrated at a small discrete set of frequency, while the spectrum of the strange non-chaotic attractor consists of much more harmonic components (see the enlarged part of spectrum, Figure 2(e)).

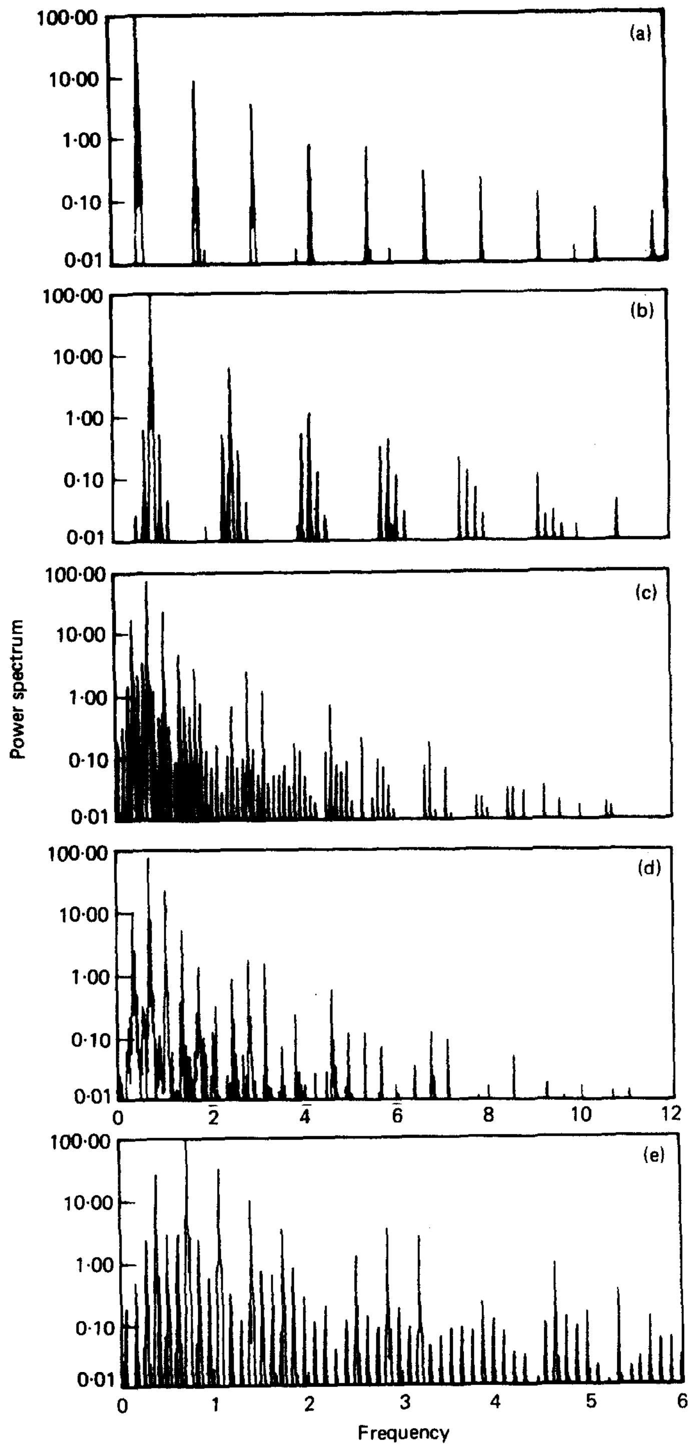

Figure 2. Power spectra of attractors shown in Figure 1; (e) enlarged part of power spectrum of Figure 2(c).

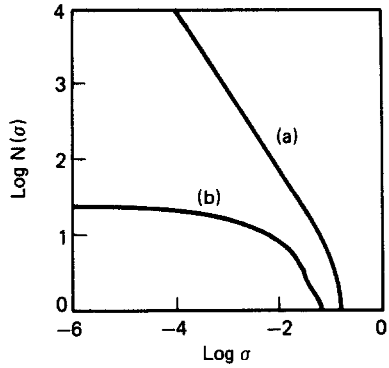

Figure 3. Spectral characteristic of attractors: (a) strange non-chaotic; (b) two-frequency quasi-periodic.

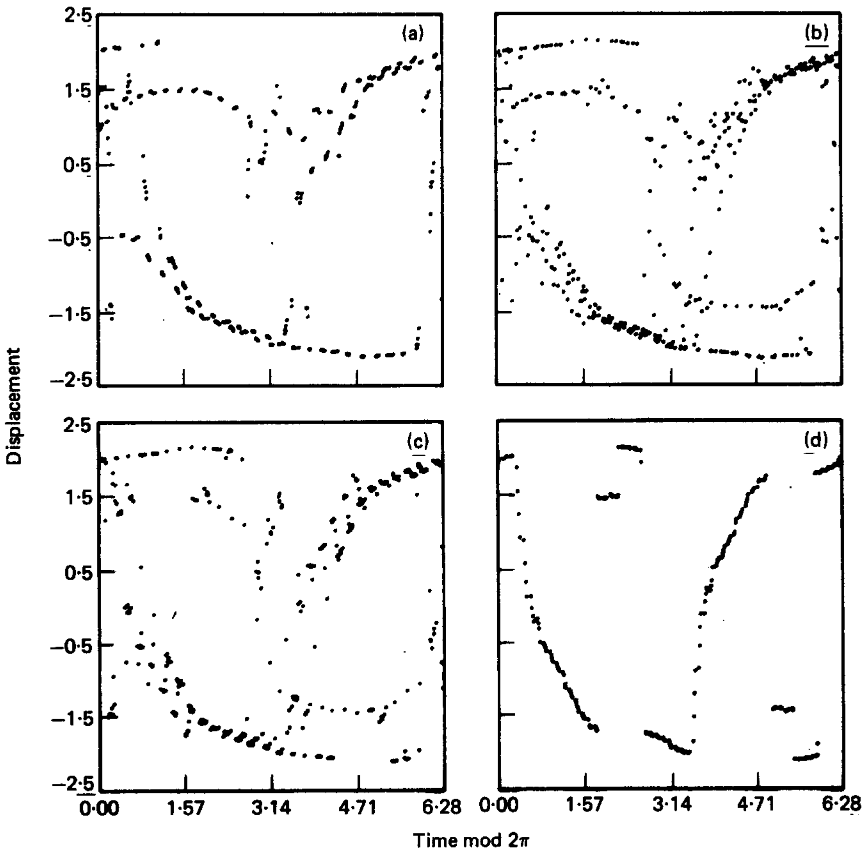

Figure 4. Evolution of attractor with increase of ω. Chaotic attractors: (a) $\omega = 1{\cdot}394$, maximum Lyapunov exponent $\lambda = 0{\cdot}023$; (b) $\omega = 1{\cdot}395$, $\lambda = 0{\cdot}022$; (c) $\omega = 1{\cdot}396$, $\lambda = 0{\cdot}017$; (k) $\omega = 1{\cdot}404$, $\lambda = 0{\cdot}033$; (*l*) $\omega = 1{\cdot}405$, $\lambda = 0{\cdot}035$. Strange non-chaotic attractors: (d) $\omega = 1{\cdot}397$; (e) $\omega = 1{\cdot}398$; (f) $\omega = 1{\cdot}399$; (h) $\omega = 1{\cdot}401$; (*i*) $\omega = 1{\cdot}402$; (j) $\omega = 1{\cdot}403$. Periodic attractor; (g) $\omega = 1{\cdot}4005$.

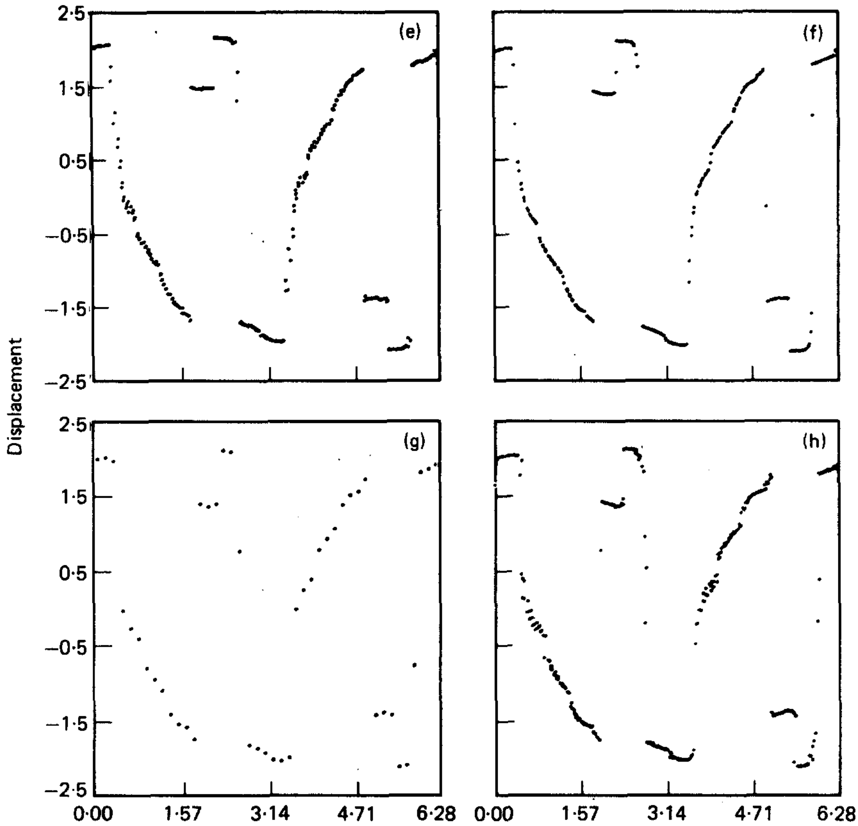

Figure 4—*continued*

In order to obtain a more quantitative characterization of the spectra of attractors we can introduce a special distribution $N(\sigma)$ which was introduced in reference [3] and defined as the number of spectral components larger than some values σ. These distributions are plotted in Figure 3. They show that strange non-chaotic attractors exhibit distinctive spectral characteristics other than those of periodic or quasi-periodic attractors. Also shown is that for strange non-chaotic attractors one has the relation $N(\sigma) \simeq \sigma^{-\alpha}$ where α is constant, while for the two-frequency quasi-periodic attractor one has $N(\sigma) \simeq \ln(\sigma^{-1})$. These relations agree with the analytical estimations of reference [3].

An example of evolution of a strange attractor is presented in Figure 4. In Figures 4(a), (b), (c) one has chaotic attractors which evolve as chaotic attractors with slight increase of ω: see Figures 4(c), (e), (f). Figure 4(g) shows a periodic attractor with a long period. With further increase of ω one observes strange non-chaotic attractors, Figures 4(h), (i), (j), and chaotic ones, Figures 4(k), (l) again.

CONCLUSIONS

A strange non-chatoic attractor is a fascinating new type of attractor which may occur for non-linear oscillators. It seems that this type of attractor is characteristic of quasi-periodically forced systems.

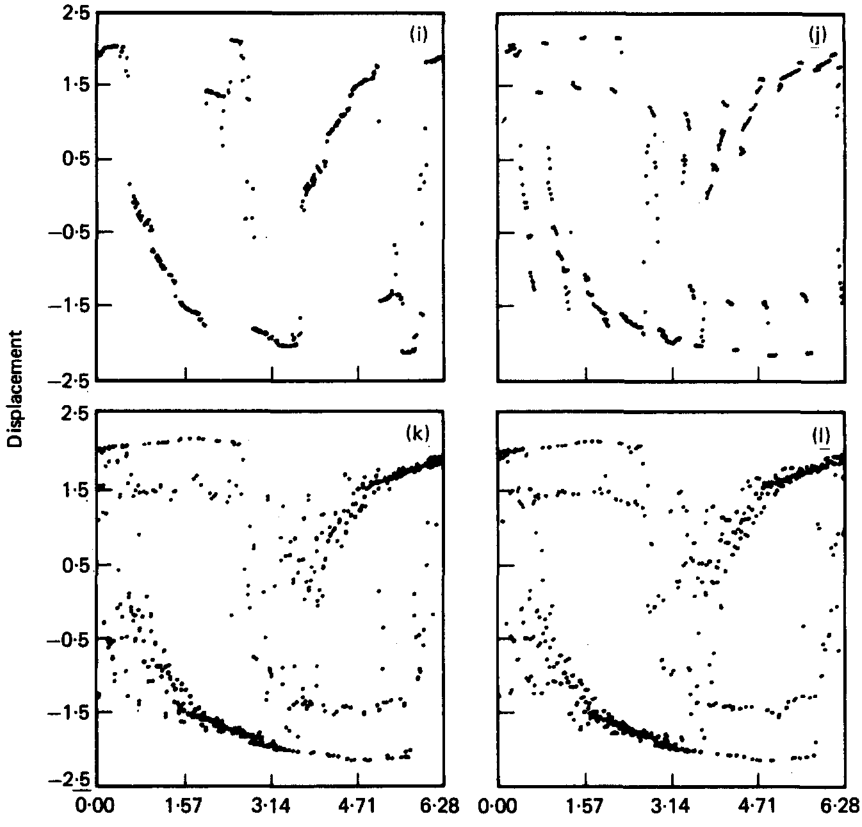

Figure 4—*continued*

ACKNOWLEDGMENT

This work was motivated by the discussion at the Euromech 242 Colloqium on Application of the Chaos Concept to Mechanical Systems. One of us (T.K.) is very thankful for a valuable discussion with P. Grassberger, H. Isömaki and F. C. Moon. We are very thankful to R. Olejnik for help in the numerical part of our investigation.

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(*Received* 4 *September* 1989)

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