ANALYTICAL CONDITION FOR CHAOTIC BEHAVIOUR OF THE DUFFING OSCILLATOR

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Received 2 November 1989; revised manuscript received 3 January 1990; accepted for publication 4 January 1990
Communicated by A.P. Fordy

A new analytical criterion is developed for the strange chaotic attractor in nonlinear oscillators which show a period-doubling route to chaos. The method is based on approximate analysis and Feigenbaum universal properties of period doubling.

It is well known that the Duffing oscillator

$$\ddot{x}+a\dot{x}+bx+cx^3=B_0+B_1\cos\Omega t \tag{1}$$

shows chaotic behaviour for certain values of the parameters [1–5]. In many cases it can be shown that the chaotic behaviour is obtained via the period-doubling bifurcation [1,2,5,6]. Recently some attempts to create an analytical criterion which allows us to estimate the domain in the system parameter space has been proposed [6,7]. The criterion by Szemplinska-Stupnicka [7] places the chaotic zone between the vertical tangent of the resonance curve of the second approximate solution and the boundary of stability of the period-two solution. In what follows the limits of stable and unstable period-doubling cascades are proposed as boundaries of the chaotic domain.

First consider the first approximate solution in the form

$$x(t)=C_0+C_1\,\mathrm{cost}(\Omega t+\vartheta)\,, \tag{2}$$

where C_0, C_1 and ϑ are constants. Substituting eq. (2) into eq. (1) it is possible to determine these constants [6,8,9]. To study the stability of the solution (2) a small variational term $\delta x(t)$ is added to eq. (2) as

$$x(t)=C_0+C_1\cos(\Omega t+\vartheta)+\delta x(t)\,. \tag{3}$$

After some algebraic manipulations, the linearized equation with periodic coefficients for $\delta x(t)$ is obtained,

$$\delta\ddot{x}+a\,\delta x+\delta x\,(\lambda_0+\lambda_1\cos\Theta+\lambda_2\cos 2\Theta)=0\,, \tag{4}$$

where

$$\lambda_0=3C_0^2+\tfrac{3}{2}C_1^2,\quad \lambda_1=6C_0C_1,$$
$$\lambda_2=\tfrac{3}{2}C_1^2,\quad \Theta=\Omega t+\vartheta\,.$$

In the derivation of eq. (4), for simplicity it was assumed without loss of generality that $b=0$. As we have a parametric term of frequency $\Omega-\lambda_1\cos\Theta$, the lowest order unstable region is that which occurs close to $\Omega/2\approx\sqrt{\lambda_0}$ and at its boundary we have the solution

$$\delta x=b_{1/2}\cos(\tfrac{1}{2}\Omega t+\phi)\,. \tag{5}$$

To determine the boundaries of the unstable region we insert eq. (5) into eq. (4), and the condition of nonzero solution for $b_{1/2}$ leads us to the following criterion to be satisfied at the boundary:

$$(\lambda_0-\tfrac{1}{4}\Omega^2)^2+\tfrac{1}{4}a^2\Omega^2-\tfrac{1}{4}\lambda_1^2=0\,. \tag{6}$$

From eq. (6) one obtains the interval $(\Omega_1^{(2)},\Omega_2^{(2)})$, within which period-two solutions exist. Further analysis shows that at Ω_2 we have a stable period-doubling bifurcation for decreasing Ω and at Ω_1 an unstable period-doubling bifurcation for increasing Ω [6]. In this interval we can consider the period-two solution of the form

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$$x(t)=A_0+A_{1/2}\cos(\tfrac{1}{2}\Omega t+\phi)+A_1\cos\Omega t\,, \qquad (7)$$

where A_0, $A_{1/2}$, A_1 and ϕ are constants to be determined. Again, to study the stability of the period-two solution we have consider a small variational term $\delta x(t)$ added to eq. (7). The linearized equation for $\delta x(t)$ has the following form:

$$\delta\ddot{x}+a\,\delta\dot{x}+\delta x\,[\lambda_0^{(2)}+\lambda_{1/2c}\cos\tfrac{1}{2}\Omega t+\lambda_{1/2s}\sin\tfrac{1}{2}\Omega t+\lambda_{3/2}\cos(\tfrac{3}{2}\Omega t+\phi)+\lambda_{1c}^{(2)}\cos\Omega t+\lambda_{1s}^{(2)}\sin\Omega t+\lambda_2^{(2)}\cos 2\Omega t]=0\,, \qquad (8)$$

where

$$\lambda_0^{(2)}=3(A_0^2+\tfrac{1}{2}A_{1/2}^2+\tfrac{1}{2}A_1^2)\,,$$
$$\lambda_{1/2c}=3A_{1/2}(2A_0+A_1)\cos\phi,$$
$$\lambda_{1/2s}=3A_{1/2}(A_1-2A_0)\sin\phi\,,$$
$$\lambda_{3/2}=3A_1A_{1/2},\quad \lambda_{1c}^{(2)}=6A_0A_1+\tfrac{3}{2}A_{1/2c}^2\cos 2\phi\,,$$
$$\lambda_{1s}^{(2)}=-\tfrac{3}{2}A_{1/2s}^2\sin 2\phi,\quad \lambda_2^{(2)}=\tfrac{3}{2}A_1^2\,.$$

The form of eq. (8) enables us to find the range of existence of a period-four solution, represented by

$$\delta x=b_{1/4}\cos(\tfrac{1}{4}\Omega t+\phi)+b_{3/4}\cos(\tfrac{3}{4}\Omega t+\phi)\,. \qquad (9)$$

After inserting eq. (9) into eq. (8) the condition of nonzero solution for $b_{1/4}$ and $b_{3/4}$ gives us the following set of nonlinear algebraic equations for Ω, $\cos\phi$ and $\sin\phi$ to be satisfied for existence:

$$(\lambda_{1/2s}+\lambda_{1s}^{(2)})-\tfrac{1}{2}(\lambda_{1/2c}-\lambda_{1c}^{(2)})\times(-\tfrac{1}{2}a\Omega+\lambda_{1/2s}-\lambda_{3/2}\sin\phi)=0\,,$$
$$(\tfrac{9}{8}\Omega^2+\tfrac{1}{2}\lambda_0^{(2)}+\lambda_{3/2}\cos\phi)-\tfrac{1}{2}(\lambda_{1/2c}+\lambda_{1c}^{(2)})(\lambda_{1s}^{(2)}+\lambda_{1/2c})=0\,,$$
$$(-\tfrac{3}{2}a\Omega-\lambda_{3/2}\sin\phi)-\tfrac{1}{2}(\lambda_{1/2c}+\lambda_{1c}^{(2)})(\lambda_{1s}^{(2)}+\lambda_{1/2c})=0\,. \qquad (10)$$

Solving eq. (10) by a numerical procedure it is possible to obtain $\Omega_1^{(4)}$ and $\Omega_2^{(4)}$, the frequencies of stable and unstable period-four bifurcations. With both stable and unstable period-two and period-four boundaries obtained from eq. (6) and eq. (10), assuming that the Feigenbaum model [10] of period doubling is valid for our system one obtains the following boundaries of the domain where chaotic behaviour may occur:

$$\Omega_1^{(\infty)}=\Omega_1^{(2)}+\frac{\Delta\Omega_1}{1-1/\delta}\,,$$
$$\Omega_2^{(\infty)}=\Omega_2^{(2)}-\frac{\Delta\Omega_2}{1-1/\delta}\,, \qquad (11)$$

where $\Delta\Omega_1=\Omega_1^{(4)}-\Omega_1^{(2)}$, $\Delta\Omega_2=\Omega_2^{(2)}-\Omega_2^{(4)}$, and $\delta=4.69...$ is the universal Feigenbaum constant. As Feigenbaum's constant is asymptotic and we extrapolate $\Omega_{1,2}^{(\infty)}$ from the period-two stability limits the values of $\Omega_{1,2}^{(\infty)}$ are approximate. The domain where chaotic behaviour can occur is proposed to be between the limits of unstable and stable period-doubling cascades, in the interval $(\Omega_1^{(\infty)}, \Omega_2^{(\infty)})$ and of course to expect chaos one must have

$$\Omega_1^{(\infty)}<\Omega_2^{(\infty)}\,. \qquad (12)$$

In fig. 1 we show the comparison of this analytical estimation of the chaotic domain and the actual (numerically found) chaotic domain obtained by Ueda [1]. Good agreement is seen. In fig. 2 we compare the actual chaotic domain with the domain obtained by the above method. Also the domain obtained by the approximate criterion of Szemplinska-Stupnicka [7] is indicated. Again our approach shows very good agreement with the actual chaotic domain and is better than other analytical estimates. In the domain σ in fig. 2 chaotic behaviour has not been found, as here $\Omega_1^{(\infty)}>\Omega_2^{(\infty)}$, and no chaotic interval estimated by our method exists.

The analytical technique presented in this paper is based on: (a) the approximate period-one, -two and -four solutions and their stability limits computed by harmonic balance method, (b) Feigenbaum's uni-

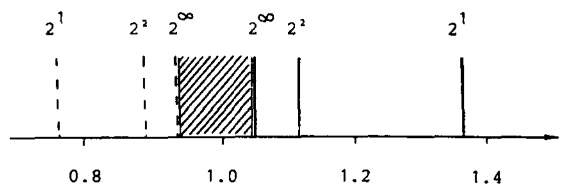

Fig. 1. Stable (solid line) and unstable (broken line) boundaries of period-two and -four bifurcations and chaotic domain for eq. (1). $a=0.77$, $b=0$, $c=1$, $B_0=0.045$, $B_1=0.16$, $\Omega_1^{(2)}=0.77$, $\Omega_2^{(2)}=1.37$, $\Omega_1^{(4)}=0.89$, $\Omega_2^{(4)}=1.12$, $\Omega_1^{(\infty)}=0.93$, $\Omega_2^{(\infty)}=1.05$. Chaotic behaviour has been found for $\Omega\in[0.94, 1.04]$ [1].

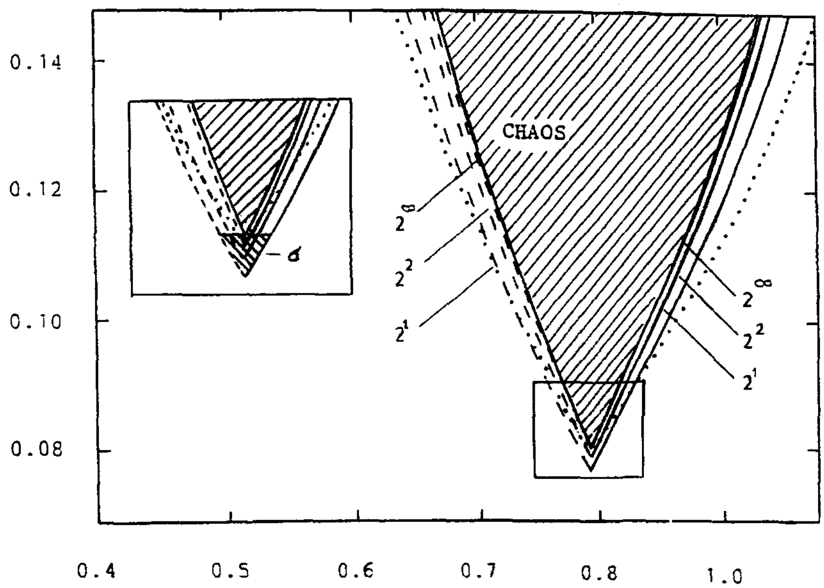

Fig. 2. Chaotic domain of the system (1), $a=0.1$, $b=0.5$, $c=0.5$, $B_0=0$, and analytical criteria, (....) criterion of ref. [7]. In the left-hand side of the figure the enlargement of the box section is shown.

versal constant for the asymptotic ratio of the stability intervals of the 2^n and 2^{n+1} periodic solution.

It can be applied to the class of oscillators for which the harmonic balance method analysis shows the possibility of period-doubling bifurcation ($\lambda_1 \neq 0$ in eq. (4)). Our method can be applied before numerical analysis to estimate the phase space intervals where strange phenomena can take place.

In future works I will try to use the above method to estimate chaotic domains of other nonlinear oscillators.

This work has been supported by the British Council. I am very thankful to J. Brindley, D.G. Knapp and E. Ponce for valuable discussions.

References

[1] Y. Ueda, Ann. NY Acad. Sci. 357 (1980) 422.
[2] Y. Ueda, J. Stat. Phys. 20 (1979) 181.
[3] S. Novak and R.G. Frehlick, Phys. Rev. A 26 (1982) 3660.
[4] J.C. Huang, Y.H. Kao, C.S. Wang and Y.S. Gou, Phys. Lett. A 136 (1989) 131.
[5] S. Sato et al., Phys. Rev. A 28 (1983) 1654.
[6] W. Szemplinska-Stupnicka and J. Bajkowski, Int. J. Nonlin. Mech. 21 (1986) 401.
[7] W. Szemplinska-Stupnicka, Ingenieur-Archiv 58 (1988) 354.
[8] T. Kapitaniak, J. Sound Vibr. 121 (1988) 259.
[9] T. Kapitaniak, Chaos in systems with noise (World Scientific, Singapore, 1988).
[10] M. Feigenbaum, J. Stat. Phys. 19 (1978) 25.