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LETTER TO THE EDITOR

Route to chaos via strange non-chaotic attractors

T Kapitaniak†§, E Ponce†|| and J Wojewoda‡§

† Department of Applied Mathematical Studies and Centre of Nonlinear Studies, University of Leeds, Leeds LS2 9JT, UK

‡ Division of Dynamics and Control, University of Strathclyde, James Weir Building, 75 Montrose St, Glasgow G1 1XJ, UK

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Abstract. The route to chaos in quasiperiodically forced systems is investigated. It has been found that chaotic behaviour is obtained after breaking of three-frequency torus, but strange non-chaotic attractors are present before three-frequency quasiperiodic behaviour occurs.

Ruelle and Takens [1] suggested that strange attractors could arise after a finite sequence of Hopf bifurcations. Later it was specified by Newhouse, Ruelle and Takens [2] that after three bifurcations strange attractors could arise.

Recently it was found that there are two types of strange attractors [3-6].

The word 'strange' refers to the geometrical structure of the attractor and an attractor which is not:

a finite set of points;

a limit cycle (closed curve);

a smooth (piecewise smooth) surface;

bounded by a piecewise smooth closed surface volume;

is called a strange attractor. An attractor is chaotic if at least one Lyapunov exponent is positive (typically nearby orbits diverge exponentially with time). From what was said above, one finds that a strange non-chaotic attractor is an attractor which is geometrically strange, but for which typical orbits have non-positive Lyapunov exponents.

Ding *et al* [4] suggested that the route to chaos from two-frequency quasiperiodicity on a T^2 torus is via three-frequency quasiperiodicity on a T^3 torus and strange non-chaotic attractors. In this letter we investigate the Ruelle-Takens-Newhouse route to chaos in the systems with two-frequency quasiperiodic forcing, i.e. from a T^2 torus to chaos. The aim of this work is to show that the route

$$\begin{aligned}\text{two-frequency quasiperiodicity} &\to \text{strange non-chaotic attractors}\\ &\to \text{three-frequency quasiperiodicity}\\ &\to \text{strange chaotic attractors}\end{aligned} \tag{1}$$

is possible and that the T^2 torus breaks before creation of the T^3 torus.

First consider van der Pol's oscillator:

$$\ddot{x}+d(x^2-1)\dot{x}+x=a\cos\omega t\cos\Omega t \tag{2}$$

§ Permanent address: Institute of Applied Mechanics, Technical University of Lodz, Stefanowskiego 1/15, 90-924 Lodz, Poland.

|| Permanent address: Department of Applied Mathematics, University of Sevilla, Avda. Reina Mercedes, 41012 Sevilla, Spain.

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where a, d, ω, Ω are constants. We considered $a = d = 5.0$, $\Omega = \sqrt{2} + 1.05$ and $\omega \in [0, 0.01]$. Equation (2) has four-dimensional phase space:

$$(x, \dot{x}, \Theta_1 = \omega t, \Theta_2 = \Omega t) \in R^2 \times S^1 \times S^1.$$

We can reduce the study of (2) to the study on an associated three-dimensional Poincaré map obtained by defining a three-dimensional cross section to a four-dimensional phase space by fixing the phase of one of the angular variables and allowing the remaining three variables that start on the cross section to evolve in time under the effect of the flow generated by (2) until they return to the cross section. If we fix the phase Θ_2, the Poincaré map is defined as a set:

$$M(t_0) = \{(x(t_n), \dot{x}(t_n), \Theta_1(t_n)) | t_n = 2\Pi n/\Omega + t_0, n = 1, 2, \ldots\}$$

where t_0 is initial time. To describe the surface of the Poincaré map we plot

$$x(t_n) \text{ against } \dot{x}(t_n).$$

Alternative surfaces can be obtained by plotting $x(t_n)$ against $\Theta_1(t_n)$, and $\dot{x}(t_n)$ against $\Theta_1(t_n) \bmod 2\Pi$. Of course, to characterise the attractor we also used Lyapunov exponents given by:

$$\lambda = \lim_{t \to \infty} \{(1/t) \ln[d(t)/d(t_0)]\}$$

where $d = \sqrt{|y^2 + \dot{y}^2|}$ and y denotes the solution of the equation variational to (2).

The winding number for orbit $x(t)$ of (2) defined by the limit

$$w = \lim_{t \to \infty} \{(\alpha(t) - \alpha(t_0))/t\}$$

where $(x, \dot{x}) = (r \cos \alpha, r \sin \alpha)$ is another quantity.

The plot of the Lyapunov exponent against ω has been shown in figure 1 (the largest non-zero exponent has been taken). For two-frequency quasiperiodic behaviour we have a negative Lyapunov exponent and winding number fulfilling the relation:

$$w = (1/n)\omega + (m/n)\Omega \tag{3}$$

where l, m, n are integers. With further decrease of ω, the Lyapunov exponent is still negative but the winding number does not satisfy relation (3) and we have the example of a strange non-chaotic attractor. When the winding number does not satisfy relation (3) and the Lyapunov exponent is zero, three-frequency quasiperiodic behaviour is present (point F_3 in figure 1). No evidence of three-frequency quasiperiodic behaviour has been found in the transition from two-frequency quasiperiodic behaviour to the strange non-chaotic attractor. This type of behaviour is present on the boundary between strange non-chaotic behaviour and chaos. In figure 2(a)–(c) the Poincaré map is shown for two-frequency quasiperiodic, strange non-chaotic and strange chaotic behaviour. In the case of both strange attractors the Poincaré map has the same structure. In figure 2(a) we observe that the strange non-chaotic attractors do not exist on the T^2 torus as this figure shows that it is broken. We investigated 1200 different attractors of system (2) and in all examples we observed the same sequence (1) and the same properties of the Poincaré map as described in figure 2.

Finally consider the map:

$$\begin{aligned} \Phi_{n+1} &= [\Phi_n + 2\Pi K + V \sin \Phi_n + C \cos \Theta_n] \\ \Theta_{n+1} &= [\Theta_n + 2\Pi\omega] \end{aligned} \tag{4}$$

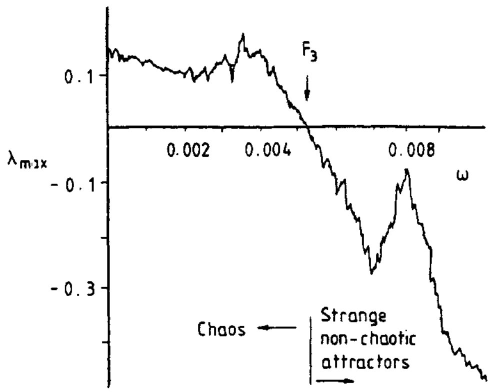
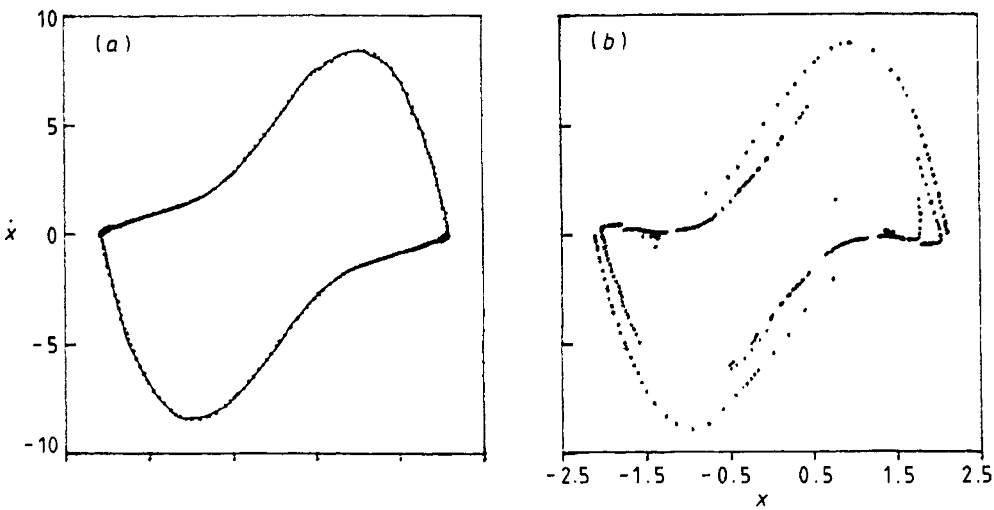


Figure 1. The largest non-zero Lyapunov exponent of equation (2) against ω; $a = d = 5$, $\Omega = \sqrt{2} + 1.05$.



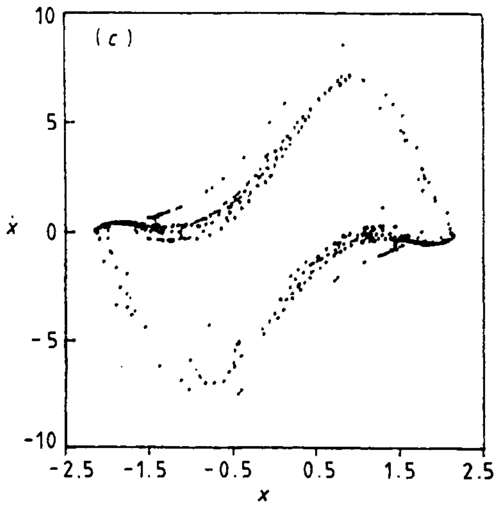


Figure 2. Poincaré maps of (2); (*a*) $\omega = 0.3$, (*b*) $\omega = 0.005$, (*c*) $\omega = 0.001$.

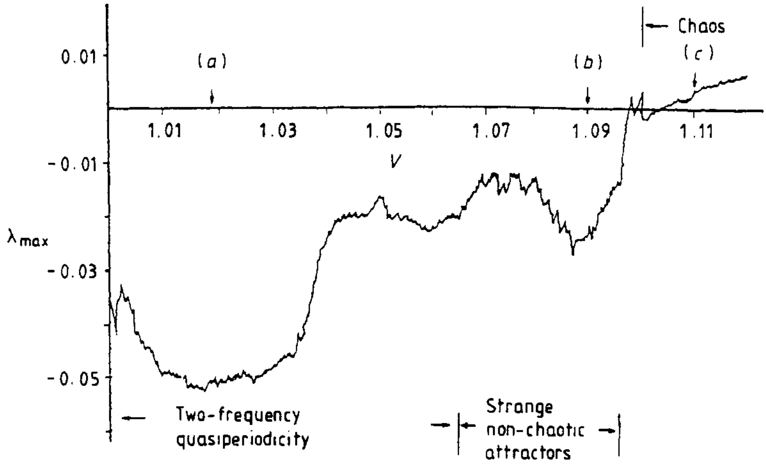


Figure 3. Lyapunov exponent (5) against V for map (4).

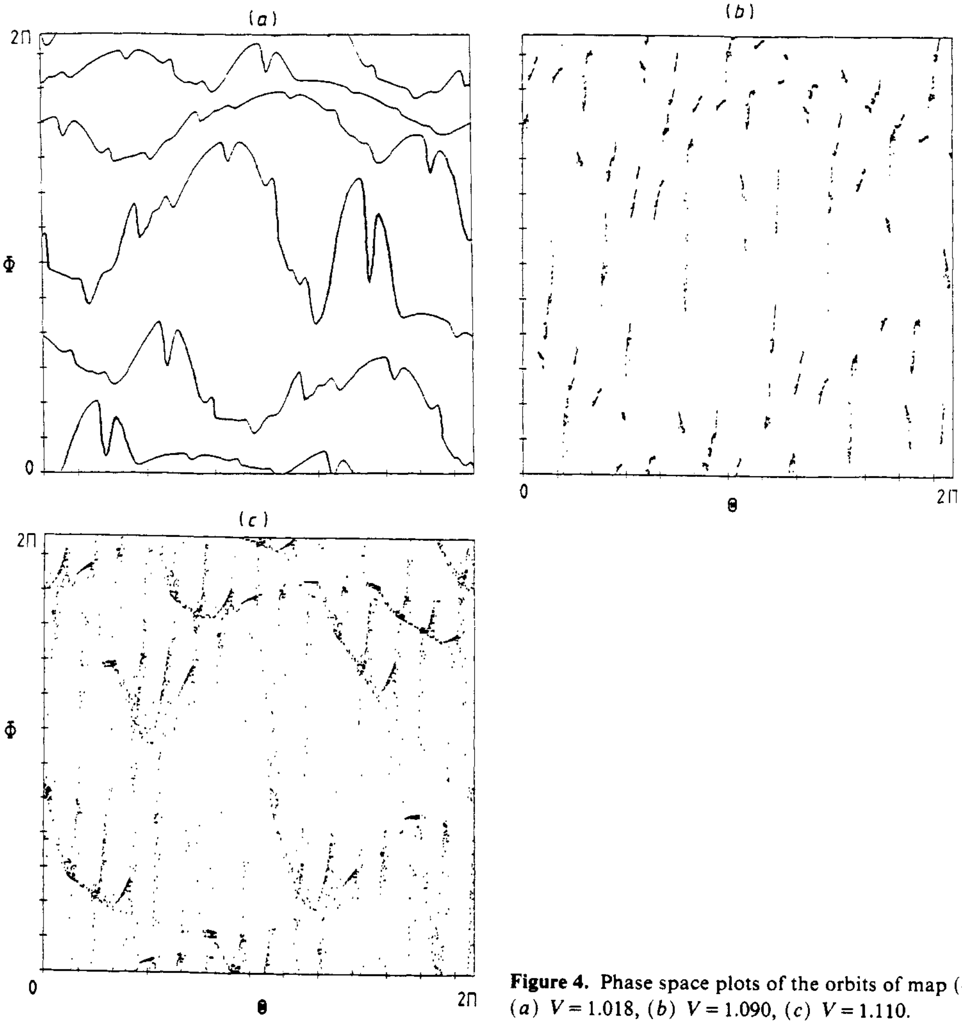


Figure 4. Phase space plots of the orbits of map (4); (*a*) $V = 1.018$, (*b*) $V = 1.090$, (*c*) $V = 1.110$.

where K, V, C and ω are constants, ω is irrational (in our numerical experiments the golden mean $\omega = (\sqrt{5} - 1)/2$ has been taken) and the square brackets indicate that modulo 2Π of the expression is taken. In [4] it was shown that the transition from quasiperiodicity to chaos leads through the region where strange non-chaotic attractors are present. In figure 3 the plot of Lyapunov exponent:

$$\lambda = \lim_{n \to \infty} (1/n) \sum_{k=1}^{n} \ln|1 + V \cos \Phi_k| \tag{5}$$

is shown against V. Up to $V \approx 1.067$ condition (3) is fulfilled, but for larger values of V we observe the transition to strange non-chaotic attractors. Strange non-chaotic attractors occur again before three-frequency quasiperiodicity. In figure 4(a)–(c) phase space plots of the orbits corresponding to two-frequency quasiperiodicity, strange non-chaotic and chaotic behaviours are shown for the V values indicated in figure 3.

To summarise, in this letter we show that the possible route to chaos in two-frequency quasiperiodically forced systems is as follows: two-frequency quasiperiodicity → strange non-chaotic attractors → three-frequency quasiperiodicity → chaos.

References

[1] Ruelle D and Takens F 1971 *Commun. Math. Phys.* **20** 167
[2] Newhouse S, Ruelle D and Takens F 1978 *Commun. Math. Phys.* **64** 35
[3] Romieras F and Ott E 1987 *Phys. Rev.* A **35** 4404
[4] Ding M, Grebogi C and Ott E 1989 *Phys. Rev.* A **39** 2593
[5] Ding M, Grebogi C and Ott E 1989 *Phys. Lett.* **137A** 167
[6] Kapitaniak T and Wojewoda J 1990 Strange nonchaotic attractors of quasiperiodically forced van der Pol's oscillator *J. Sound and Vibration* in press