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LETTERS TO THE EDITOR

TRANSITION TO CHAOS IN A GENERALIZED VAN DER POL'S EQUATION

It is well known that the generalized van der Pol equation with an external periodic force

$$\frac{d^2x}{dt^2} - a(1-x^2)\frac{dx}{dt} + bx + cx^3 = d \sin \Omega t \tag{1}$$

shows chaotic behaviour both in the case $c = 0$ (for example $a = d = 5$, $b = 1$ and $\Omega = 2{\cdot}466$ (see reference [1])) and $b = 0$ (for example $a = 0{\cdot}2$, $c = 1$, $d = 17$ and $\Omega = 4$ (see reference [2])).

What follows is an analysis of the influence of the small cubic term cx^3 on the chaotic behaviour of equation (1) which can be characterized by the largest one-dimensional Lyapunov exponent. We set $a = d = 5$, $b = 1$ and $\Omega = 2{\cdot}466$. This problem has some practical significance as in many practical problems one has to deal with restoring force of the type $bx + cx^3$. For a weakly non-linear system (c small) usually linearizes this relation.

For numerical studies equation (1) is rewritten as a system of first order differential equations with $y \doteq dx/dt$. A modified Runge-Kutta method of the fourth order was used for the numerical simulations. The calculation step was $\pi/(100\Omega)$.

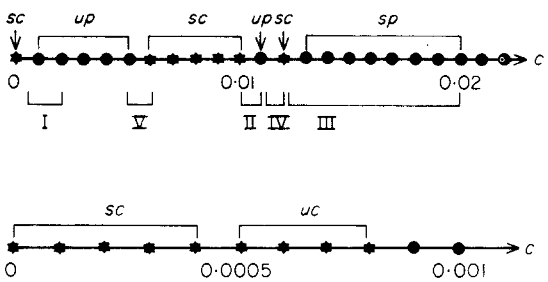

Figure 1. Chaotic and periodic attractors on the parameter c line; ★, chaotic; ●, periodic; *uc*, unsymmetrical chaotic; *sc*, symmetrical chaotic; *up*, unsymmetrical periodic; *sp*, symmetrical periodic; I-V, zones where transitions from chaotic to periodic behaviour (or *vice versa*) take place.

The behaviour of system (1) in its dependence on c is shown in Figure 1. The chaotic behaviour is marked with a star (★) and the periodic behaviour is marked with a dot (•). First we examined the c interval [0, 0·02] in steps of 0·001. Then in searching for routes to chaos we used smaller steps. It is interesting that in this small interval there are two types of periodic attractors, and two types of chaotic attractors. Examples of periodic unsymmetrical attractors are shown in Figures 2(a), 2(b) and 3(a), 3(b) and marked by *up* in Figure 1. The periodic symmetrical attractors are shown in Figures 4(a), 4(b) and they are marked by *sp* in Figure 1. An example of a chaotic unsymmetrical attractor (marked by *uc* in Figure 1) is shown in Figure 2(c), and examples of chaotic symmetrical attractors (*sc* in Figure 1) are shown in Figures 3(c) and 4(c). When $c = 0$ one has a chaotic symmetrical attractor.

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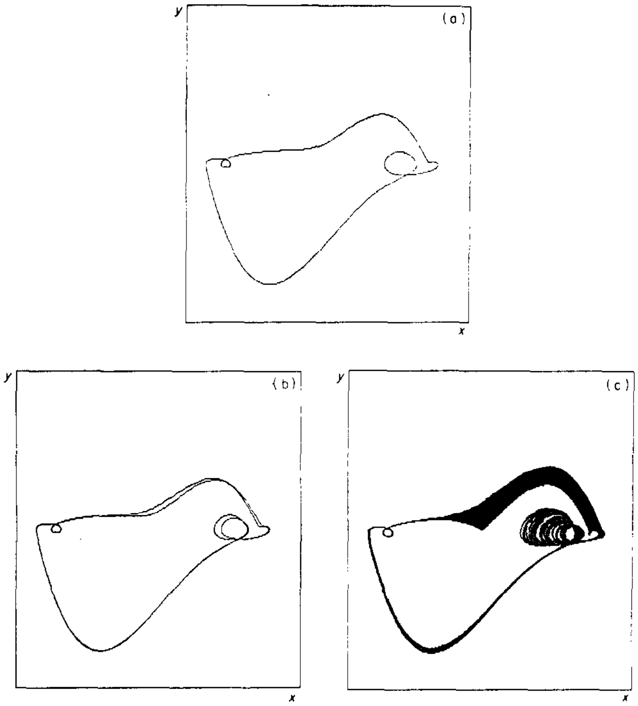

Figure 2. Transition to chaos in region I. Phase portraits: (a) $c = 0{\cdot}002$; (b) $c = 0{\cdot}001$; (c) $c = 0{\cdot}0005$.

Two different routes to chaos are found. The first one starts with a periodic unsymmetrical attractor of the type shown in Figures 2(a) and 3(a) with a small and a large loop. With decreasing c a period-doubling bifurcation takes place. These bifurcations are visible on the phase portrait by the creation of new large loops (Figures 2(b) and 3(b)). Finally this route can lead to chaotic symmetrical or unsymmetrical attractors (Figures 2(c), 3(c) and 4(c)). This type of route to chaos takes place in regions I and II.

For an unsymmetrical chaotic attractor ($c = 0{\cdot}0005$, Figure 2(c)) the attractor evolves to a symmetrical one after a small decrease in c.

For the route shown in Figures 3(a)–3(c) a chaotic unsymmetrical attractor between periodic unsymmetrical and chaotic symmetrical attractors is not found.

The second route starts from a symmetrical periodic attractor shown in Figure 4(a) and it takes place in the region III in Figure 1. With decreasing c one observes first that the structure of the small loop is changing (Figure 4(b)) and then the system suddenly undergoes a transition to the chaotic attractor shown in Figure 4(c).

Similar sudden jumps from chaotic symmetrical attractors to periodic unsymmetrical attractors take place in regions IV and V shown in Figure 1.

As mentioned above, system (1) shows chaotic behaviour for $a = d = 5$, $b = 1$, $\Omega = 2{\cdot}466$ and $c = 0$. When we take into account the non-linearity cx^3 one would expect that this non-linearity would strengthen the chaotic behaviour. However, we find that with a small cubic term (for example, $c = 0{\cdot}013$) the chaotic behaviour disappears. Even for smaller values of c there exist windows with periodic attractors.

To summarize we find that the chaotic behaviour of system (1) is very sensitive to the small additional cubic non-linearity cx^3. When chaotic behaviour is expected in any experiment, our results show that we cannot apply linearization of cx^3 since it can change

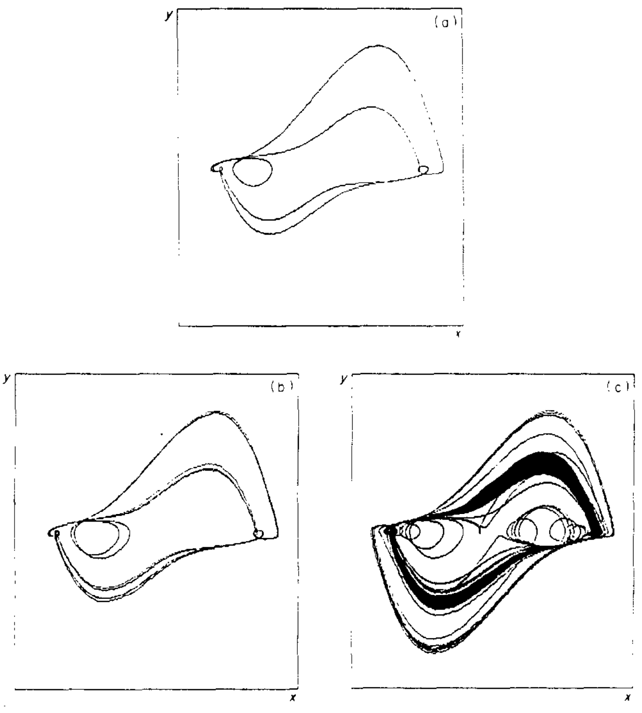

Figure 3. Transition to chaos in region II. Phase portraits: (a) $c = 0{\cdot}011$; (b) $c = 0{\cdot}0109$; (c) $c = 0{\cdot}0108$.

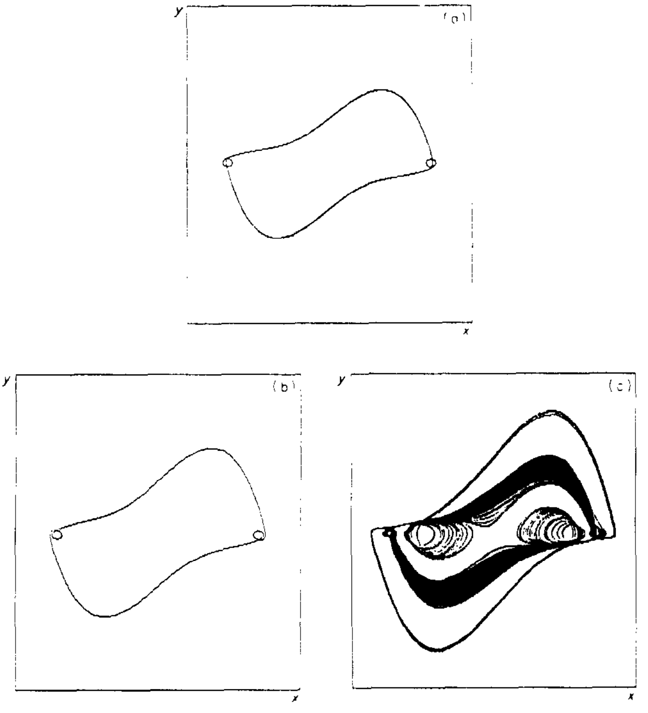

Figure 4. Transition to chaos in region III. Phase portraits: (a) $c = 0{\cdot}02$; (b) $c = 0{\cdot}0128$; (c) $c = 0{\cdot}012$.

the solution qualitatively. This is due to the fact that the "damping term" is non-linear. Consequently, both non-linear terms must be taken into account.

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