Transition to hyperchaos in coupled generalized van der Pol equations

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It has been shown that two coupled generalized van der Pol equations can show hyperchaotic behaviour, i.e. the first two one-dimensional Lyapunov exponents are positive. The scaling law for transition from chaos to hyperchaos based on the properties of the Poincaré map has been found. For fixed parameter values we also found that different behaviours of the system, such as limit cycles, chaos and hyperchaos, can coexist.

In recent years much interest has been expressed with regard to chaotic phenomena of nonlinear systems. In particular, nonlinear first-order one-dimensional mappings, such as the logistic mapping, three-dimensional autonomous systems of first-order ordinary differential equations, such as the Lorenz model and conservative Hamiltonian systems with two degrees of freedom, such as the Hénon–Heiles model, have been studied. The first one-dimensional Lyapunov exponent can be positive and the other ones are zero or negative. If we have a positive Lyapunov exponent, then in almost all cases one can conjecture that there is some invariant measure behind it, with positive Hausdorff dimension. So the system has positive metric entropy, i.e. real chaos.

For certain first-order nonlinear two-dimensional mappings with bounded trajectories we can find that the two one-dimensional Lyapunov exponents can be positive for certain initial values and parameter values. The most studied example is the coupled logistic equation [1–5]. For a first-order autonomous system of ordinary differential equations we also can find systems with two positive one-dimensional Lyapunov exponents. Such systems have been discussed by Rossler [6] and Matsumoto et al. [7]. Thus we define hyperchaos as a chaotic attractor with more than one positive one-dimensional Lyapunov exponent. In other words, the dynamics expands not only small line elements, but also small area elements, thereby giving rise to a thick chaotic attractor. In a conservative Hamiltonian system with three degrees of freedom we can also find hyperchaos (in this case also called strong chaos) [8].

In this paper we investigate the two coupled generalized van der Pol equations

$$\ddot{x}-a(1-x^2)\dot{x}+x^3=b\,(\sin\omega t+y)\,,$$
$$\ddot{y}-a(1-y^2)\dot{y}+y^3=b\,(\sin\omega t+x)\,, \qquad (1)$$

where a, b and ω are constant, and we focus our attention on the possibility of hyperchaotic behaviour and on the route from chaos to hyperchaos. Al-

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though it may seem that it is not difficult to find a system with two positive Lyapunov exponents by coupling two chaotic oscillators, our numerical examples show that it is not so easy. In many examples we found that two coupled oscillators (for such system parameters that each of them shows chaotic behaviour without coupling) show exponential divergence of the nearby trajectories only in one direction (one positive Lyapunov exponent) or even regular behaviour [9,10].

We fixed $a=0.2$, $\omega=4.0$ and considered different b. After a transposition $x_1=x$, $x_2=\dot{x}$, $x_3=y$, $x_4=\dot{y}$, $x_5=\omega t$ we can rewrite eqs. (1) as an autonomous system of five ordinary differential equations:

$$\begin{aligned}
\dot{x}_1 &= x_2 , \\
\dot{x}_2 &= a(1-x_1^2)x_2 - x_1^3 + b\,(\sin x_5 + x_3) , \\
\dot{x}_3 &= x_4 , \\
\dot{x}_4 &= a(1-x_3^2)x_4 - x_3^3 + b\,(\sin x_5 + x_1) , \\
\dot{x}_5 &= \omega , \qquad (2)
\end{aligned}$$

where $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^4 \times \mathbb{S}^1$.

In our numerical investigations we used the ODE procedure of Wolf et al. [11] to compute the spectrum of Lyapunov exponents. Eqs. (2) have been integrated by the Runge–Kutta method with time step $\pi/100\omega$. Part of the computations have been performed using Yorke's package DYNAMICS [12].

The structure of the phase space indicates that one of the Lyapunov exponents must be zero. As well as the typical chaotic attractors (with one positive Lyapunov exponent), it has been possible to find attractors with two positive Lyapunov exponents (+, +, 0, −, −) [9]. A few examples of hyperchaotic attractors are shown in table 1. To our knowledge these are the first examples of hyperchaos in systems of coupled oscillators. In the case of this attractor two positive Lyapunov exponents indicate exponential spreading within the attractor in directions transverse to the flow and negative exponents indicate exponential contraction onto the attractor. Under the action of such a flow, phase space volumes evolve in the way schematically shown in fig. 1.

Projections of Poincaré maps onto the plane x_1–x_2 are shown in fig. 2. In fig. 2a we show the Poincaré map of chaotic behaviour ($b=6.0$), while in fig. 2b the Poincaré map describes hyperchaotic behaviour

Table 1
Examples of hyperchaotic attractors: $x(0)=1.0$, $\dot{x}(0)=y(0)=\dot{y}(0)=0$.

b	Lyapunov exponents				
6.20	0.58	0.10	0	−0.46	−0.87
6.25	0.61	0.15	0	−0.46	−0.87
6.50	0.58	0.16	0	−0.57	−0.87
6.75	0.65	0.19	0	−0.62	−0.92
7.00	0.69	0.23	0	−0.66	−0.94
7.25	0.71	0.24	0	−0.70	−0.95
7.50	0.72	0.27	0	−0.73	−0.96
7.75	0.73	0.25	0	−0.74	−0.96
8.00	0.73	0.24	0	−0.74	−0.97
8.25	0.75	0.23	0	−0.73	−0.99

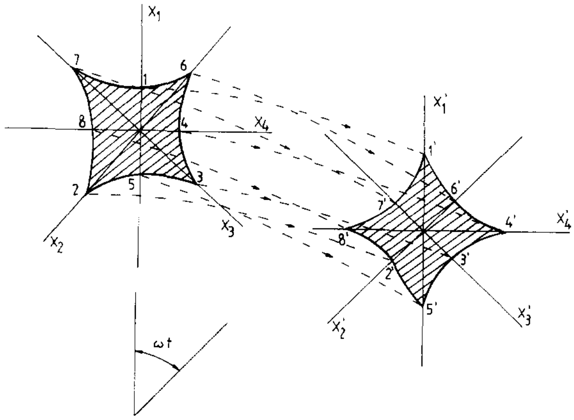

Fig. 1. Schematic evolution of phase space volume under the action of hyperchaotic flow.

($b=7.0$). In both cases the initial transient of $10^4 T$ ($T=2\pi/\omega$) has been cut.

Generally in systems with chaotic and hyperchaotic behaviour there is no qualitative difference between the Poincaré maps in both cases [9]. In system (1) we succeed in describing the transition to hyperchaos based on the x_1–x_2 projection of the Poincaré map.

If we compare fig. 2a with fig. 2b we find that in a chaotic case we have four domains (denoted by A in fig. 2c) where the concentration of points is greater than in other parts of the map (denoted by B in fig. 2c). In the hyperchaotic example we have not such domains. The successive Poincaré map point of the chaotic trajectory stays in one of the domains A for a relatively long time and escapes from them to do-

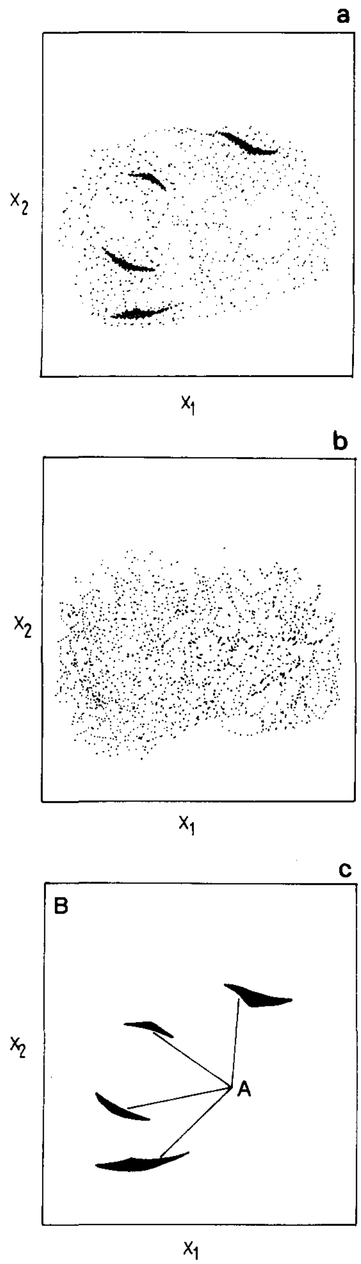

Fig. 2. x_1–x_2 projections of Poincaré map: (a) chaotic behaviour $b=6.0$, (b) hyperchaotic behaviour $b=7.0$, (c) domains A and domain B, ($b=6.0$).

main B for a much shorter time after which it comes back to A. Numerically the domains A have been estimated in the following way. After cutting the transient we observed the Poincaré map for a number of periods T. The successive point of the Poincaré map is in domains A if its distance from any of the previous points is not larger than the maximum distance between a pair of previous points. We can define the following symbolic dynamics:

1: the point of the Poincaré map is in domains A,

0: the point of the Poincaré map is in domain B.

For example for $b=5.2$ we have the following sequence of symbols,

$$\underset{(10^4+120)T}{(1111...1)}\ \underset{34T}{(0...0)}\ \underset{(10^4+78)T}{(111...1)}\ \underset{43T}{(0...0)}\ \ldots .$$

The numbers given under each sequence in parentheses indicate the number of periods T for which each sequence takes place.

Let R be an average number of periods T for which the trajectory stays in domains A. If we are increasing the value of b towards the value $b=6.17$ which is the boundary of hyperchaos we observe that R decreases (see fig. 3).

From fig. 3 we have the following scaling law for transition from chaos to hyperchaos,

$$R \sim b^{-\alpha}, \tag{3}$$

where $\alpha \approx 1.52$.

A similar relation can be obtained by the investigation of other projections of the Poincaré map.

For the same parameter values and different initial values it is possible to arrive at different attractors. The example of coexisting attractors for the same parameter values and different initial condi-

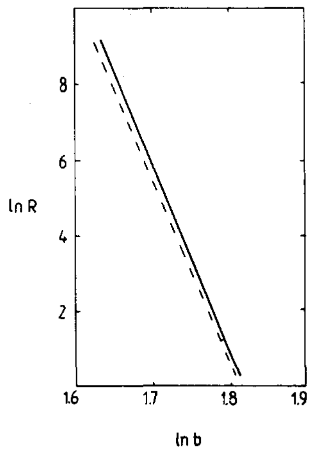

Fig. 3. R versus b in double logarithmic scale: (——) x_1–x_2 projection, (---) x_1–x_3 projection.

Table 2
Examples of coexisting attractors: $b=6.0$.

Initial values				Lyapunov exponents					Periodicity
$x(0)$	$\dot{x}(0)$	$y(0)$	$\dot{y}(0)$						
1	0	0	0	0.48	0	−0.01	−0.25	−0.83	∞
−0.546875	−3.503052	0	0	0.49	0.03	0	−0.35	−0.83	∞
0.025391	0	0	0	0	−0.43	−0.43	−0.43	−0.44	1
−0.022186	0	0	0	0	−0.48	−0.48	−0.48	−0.49	3

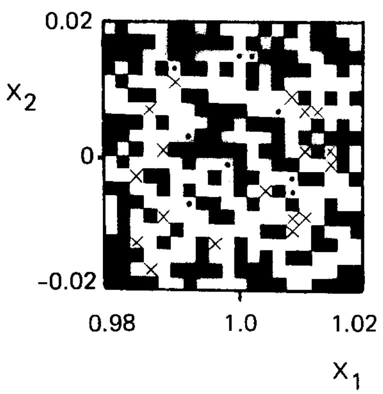

Fig. 4. Fractal boundary of basins of attraction of coexisting attractors: $b=7.0$, (□) chaos, (■) hyperchaos, (×) $3T$ periodic, (•) T periodic.

tions is shown in table 2. For these parameter values four types of attractors: hyperchaotic, chaotic, and two types of periodic attractors, have been found. The boundary between its basins of attraction is fractal. An example of it is shown in fig. 4.

To summarize we found that the system of coupled generalized van der Pol equations can show hyperchaotic behaviour. Hyperchaotic attractors coexist with chaotic and periodic attractors. The transition from chaos to hyperchaos can be described by the properties of the Poincaré map. The scaling law for this transition has been found.

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