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On Strange Nonchaotic Attractors and their Dimensions

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Abstract—The work focuses attention on strange but nonchaotic behaviour of dynamic systems as a relatively new phenomenon distinct from the usual strange chaotic one. Several characterization techniques associated with Poincaré sections, Lyapunov exponents, capacity and information dimensions are used. It is shown that based on a single time series it is virtually impossible to distinguish between strange chaotic and strange nonchaotic behaviour even using Lyapunov exponents techniques. However a combination of the capacity dimension, Kaplan–Yorke conjecture and a recent model for strange nonchaotic behaviour due to El Naschie seems to give some hope and direction for resolving this problem.

1. INTRODUCTION

In the last decade or so, much attention was given mainly to a class of dissipative dynamical systems that exhibit strange chaotic behaviour [1–3]. This behaviour and the associated attracting sets were found in numerical experiments [3, 4] as well as in actual experimental systems [5, 6].

Recently, due to the dedicated efforts of Ott, Grebogi, Yorke and their associates at Maryland a new class of strange attracting sets were found which look topologically strange but are nonchaotic [7–9]. In other words these strange nonchaotic attractors have fractal structure but typical nearby orbits do not diverge exponentially with time.

Since both attractors look visually very similar as can be seen from the Poincaré maps of Figs. 1(a,b) and 2(a,b), the numerical values of things like fractal dimensions, information dimensions and Lyapunov exponents are the only way which can allow us to make any distinctions.

In this work, we present some numerical experiments showing that it is virtually impossible to distinguish between strange chaotic and strange nonchaotic deterministic behaviour based on the estimation of Lyapunov exponents from one time series. Second, we outline a possible method for making a distinction based on Kaplan–Yorke conjecture. Finally we discuss a model proposed by El Naschie as a prototype for strange nonchaotic behaviour [18].

We will be mainly concerned here with a quasi-periodically forced Van-der-Pol's equation

$$\ddot{x} - a(1 - x^2)\dot{x} + x = d\cos\omega t\cos\Omega t \tag{1}$$

and a quasi-periodically forced pendulum

$$\ddot{x} + a\dot{x} + b\sin x = d + c(\cos\omega t + \cos\Omega t) \tag{2}$$

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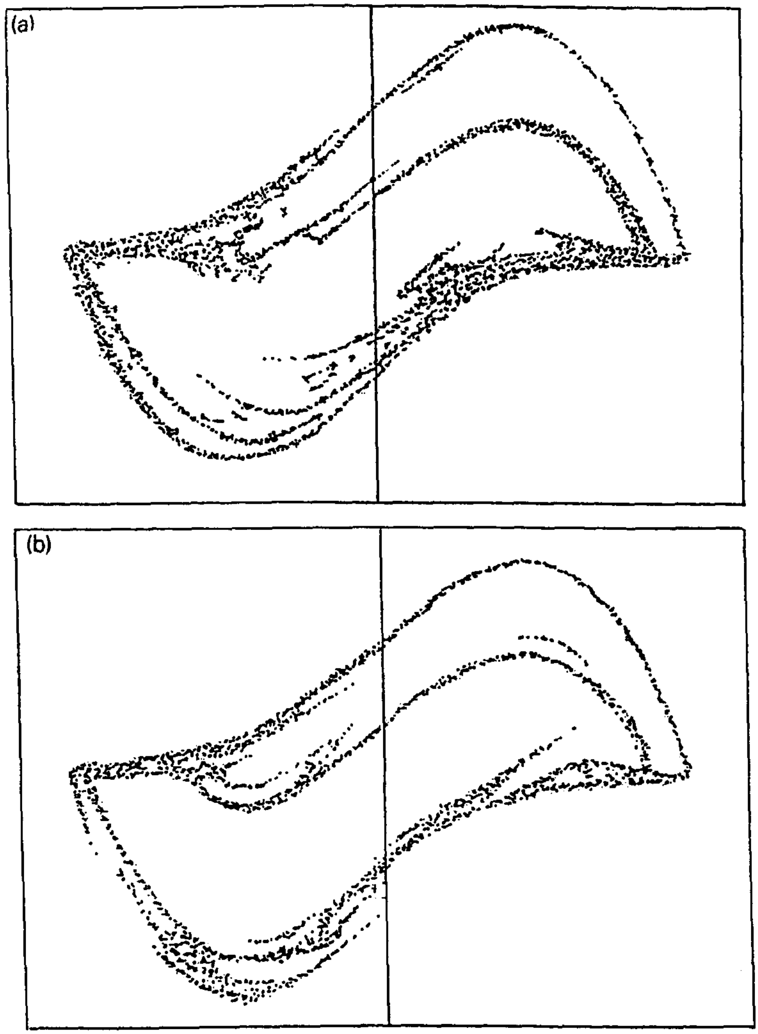

Fig. 1. The Poincaré maps of the quasi-periodically forced Van der Pol's equation (1): $a = 5.0$, $d = 5.0$, $\Omega = \sqrt{2} + 1.05$; (a) strange chaotic attractor $\omega = 0.002$, the largest nonzero Lyapunov exponent is $\lambda = 0.1183$, (b) strange nonchaotic attractor $\omega = 0.006$, $\lambda = -0.1213$.

where a,b,c,d,w and Ω are constants. Furthermore ω and Ω are incommensurate. Both equations (1) and (2) span each a four-dimensional phase space given by

$$(x_1 = x,\ x_2 = \dot{x},\ x_3 = \Theta_1 = \omega t,\ x_4 = \Theta_2 = \Omega t) \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1.$$

The study of these systems can now be reduced to the associated three-dimensional Poincaré maps obtained by defining a three-dimensional cross-section to the four-dimensional phase space by fixing the phase of one of the angular variables and following the remaining three variables as they evolve in time starting on the surface until they return to intersect the cross-section again. If the phase $x_4 = \Theta_2$ is fixed then our map would be defined as the set

$$\mathrm{M}(t_0) = \{(x_1(t_n),\ x_2(t_n),\ x_3(t_n))|_{t_n} = 2\pi n/\Omega + t_0,\ n = 1, 2 \ldots\}$$

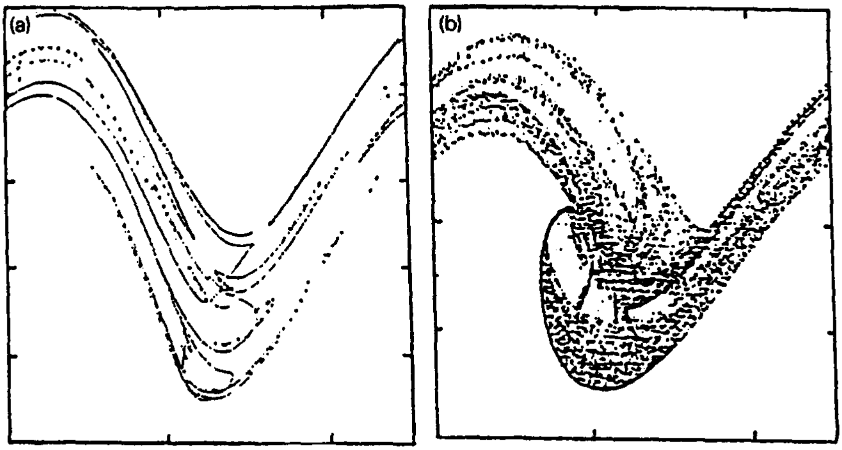

Fig. 2. The Poincaré maps of the quasi-periodically forced pendulum equation (2), $b = 1.0$, $c = 1.1$, $d = 1.33$, $\omega = 0.5 - (\sqrt{5} - 1)/4$, $\Omega = 0.5 + (\sqrt{5} - 1)/4$; (a) strange chaotic attractor $a = 0.5$, (b) strange nonchaotic attractor $a = 3.0$.

where t_0 is the initial time. The surface of the map is thus described by plotting $x_1(t_n)$ against $x_2 = \dot{x}(t_n)$. Alternatively we could have plotted $x_1(t_n)$ against $x_3(t_n)$ or $x_1(t_n)$ against $x_3(t_n)$.

2. LYAPUNOV EXPONENTS AND DIMENSIONS OF ATTRACTOR

For systems for which the equations of motion are explicitly known and the linearized equations exist there is a straightforward technique for computing a complete Lyapunov spectrum [10, 11].

For most experimental systems, however, the equations of motion are usually unknown [12] or are in a form for which linearized equations do not exist [6, 13]. In such cases, Lyapunov exponents are estimated based on the monitored long-term time series. First the attractor is reconstructed by a well-known technique with delay coordinates [14]. The reconstructed attractor though defined by a single trajectory can provide points that may be considered to lie on different trajectories. It has been shown that in many cases this attractor possesses a Lyapunov spectrum identical to the original attractor [14].

The technique of estimating the Lyapunov exponents based on the reconstructed attractor gives good results when we have at least one positive Lyapunov exponent in the spectrum.

For equations (1) and (2) the linearized equations exist and we can compute the Lyapunov exponents directly using the formula

$$\lambda = \lim_{t\to\infty} d(t)/d(0) \tag{3}$$

where $d = (y^2 + \dot{y}^2)^{1/2}$, while y is a solution of the linearized equation. In our case we have four Lyapunov exponents. Two of them are trivial in the sense that they are identically zero by virtue of the two forcing frequencies. Let the Lyapunov exponents λ_{1-4} be ordered by size

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$$

then, we can differentiate between three possibilities for equations (1) and (2):

(a) If $\lambda_1 = \lambda_2 = 0 \geq \lambda_3 \geq \lambda_4$ then we have a two-frequency quasi-periodic or a strange nonchaotic attractor.

(b) If $\lambda_1 = \lambda_2 = \lambda_3 = 0 > \lambda_4$ then we have a three-frequency quasi-periodic attractor.
(c) Finally if $\lambda_1 > 0$ then we have a strange chaotic attractor.

The method of distinguishing two- and three-frequency quasi-periodic attractors has been given in [7].

Having calculated the Lyapunov exponents one can now introduce the Lyapunov dimension

$$d_L = j + \frac{\sum_{i=2}^{j} \lambda_i}{|\lambda_{j+1}|} = j + k$$

where j is the largest index of the Lyapunov exponents for which the sum $\sum_{i=2}^{j} \lambda_i$ is nonnegative, Kaplan and Yorke [1, 11] suggested the equality of Lyapunov dimension d_L and information dimension d_I. This conjecture which was rejected initially has since been shown to be correct for countless examples and has found fruitful application on many occasions although we must stress that it does not work smoothly for certain systems. Roughly speaking the conjecture is correct only as long as the invariant set has the structure of a Cartesian product of an Euclidian manifold and a single Cantor set. We may note here that the Kaplan–Yorke formula has only one fractional part which corresponds to a single Cantor set because we have always $k < 1$.

Now we turn our attention to the information dimension of an attractor [1, 2, 11]. This dimension which unlike the capacity dimension is not a metric dimension is defined as [16]

$$d_I = \lim_{\varepsilon \to 0} I(\varepsilon)/\ln(1/\varepsilon)$$

where we are supposing that the attractor is covered by cubes formed from a Cartesian grid of spacing ε in the phase space and $I(\varepsilon)$ is given by

$$I(\varepsilon) = -\sum_{i=1}^{N(\varepsilon)} P_i \ln(P_i)$$

Here P_i is the measure of the attractor in the ith cube of the cover and $N(\varepsilon)$ is the number of cubes. In an actual calculation p_i can be estimated as the fractions of time that an orbit spends in the ith cube.

Now before we go any further we have to make it aboundantly clear that we are associating the word dimension with three different things which have to be kept clearly apart in order to avoid confusion. There is first the dimension of the phase space of the dynamical system which is equal to the number of the equivalent first order ordinary differential equations. In our case of a two-frequency forced oscillator this is clearly 4. Second we may be talking about the dimension of the whole attracting set embedded in the four-dimensional phase space. Finally we will be talking about the dimension of an attractor on the surface of cross-section which is the dimension of the Poincaré map and is thus always smaller or at best equal to the dimension of the whole attractor.

Coming back to our problem, in the presence of the two frequencies quasi-periodic forcing our attracting set must be naturally at least two dimensional. If the attractor is to be chaotic then we must have at least one unstable direction, i.e. one positive Lyapunov exponent. By virtue of Kaplan–Yorke conjecture this implies that any chaotic attractor in our four-dimensional system will have an information dimension of at least three. In fact according to Kaplan–Yorke conjecture an attractor with $d_I = 2$ must be nonchaotic. On the other hand an attractor with $d_I = 2$ is obviously neither a point nor a limit cycle. Consequently if it is not quasi-periodic then it must be indeed a new phenomenon, a 'strange' strange nonchaotic attractor.

One should not be misled into thinking that an integer as a dimension precludes the existence of a fractal structure. Actually a fractal curve with integer as a dimension was for El Naschie [18], the starting point of his peano-like dynamics idea which will be discussed later on. Putting it in another way one could say that Kaplan–Yorke conjecture has anticipated the possible existence of strange nonchaotic attractors before its actual numerical discovery which has also been verified experimentally quite recently [17].

Applying the preceding conclusions to a Poincaré map of our system, all what we need is to reduce all dimensions by one. Consequently for a strange chaotic behaviour the dimension of the attractor on the surface of the cross section should be at least $d_I \cong 2$ while for a strange nonchaotic attractor it should be $d_I \cong 1$. We have written here approximately because in general all our results are obtained from numerical calculations, using box counting methods performed not on the real three dimensional cross-section but on its two dimensional projection. This, in addition to the inherent inaccuracy of the box counting and similar methods, makes it very difficult to find for instance the theoretical value $d_I = 1$ for a strange nonchaotic attractor.

Having said that, the situation for the capacity dimension is considerably better. As well known this dimension is defined as [19]

$$d_c = \lim_{\varepsilon \to 0} \ln N(\varepsilon)/\ln(1/\varepsilon).$$

It is a metric dimension which is usually marginally but strictly larger than d_I. However for strange nonchaotic attractors, it was shown by Ding *et al.* [15] to be substantially larger. In fact El Naschie concluded on the basis of his prototype modification of Smale's horseshoe that a strange nonchaotic attractor in a two-frequency quasi-periodic system is characterized by $d_c \to 2$ and $d_I \to 1$ on the Poincaré map [18].

3. NUMERICAL EXPERIMENTS WITH STRANGE CHAOTIC AND STRANGE NONCHAOTIC ATTRACTORS

Having discussed the main theoretical background of chaotic and nonchaotic strange sets we can now discuss some of the numerical results obtained for equations (1) and (2). Thus we have computed the Lyapunov exponents twice, once from equation (3) and another time from time series based on Wolf *et al.*'s algorythm [10]. The results are compared in Table 1. From this table we see clearly that the Lyapunov exponent calculated from the explicitly known differentiable equations enables us to distinguish between strange chaotic and strange nonchaotic attractors. This distinction is however not possible based on the Lyapunov exponents estimated from a single time series. This is so because using this method we obtain positive Lyapunov exponents for chaotic and nonchaotic strange behaviour alike.

The difference between chaotic and nonchaotic attractors can be made quite visible when we plot the Poincaré map for two nearby trajectories and connect successive points of both trajectories as shown in Fig. 3. In the case of strange nonchaotic behaviour it is clearly visible that both trajectories remain close together as in Fig. 3(a) while in the chaotic case sensitive dependence on initial conditions is visibly documented in Fig. 3(b).

We have made capacity and information dimension calculations [16] on the projection $x - \dot{x}$ of the Poincaré map. Every time a maximum of 2000×2000 equally sized grid boxes were used to cover the attractor. The results are presented in Table 2. All dimensions were estimated from the slope of the logarithmic plot. For the capacity dimension this is $\ln N(\varepsilon) - \ln(1/\varepsilon)$ and for the information dimension this is $I(\varepsilon) - \ln(1/\varepsilon)$. An example of such a plot is shown in Fig. 4.

In case of the strange nonchaotic attractor of Fig. 1(b), least square fit gave $d_I \cong 1.45$

Table 1. Comparison between the values of the Lyapunov exponents computed from formula (3) and that estimated from time series

Equation	a	b	c	d	ω	Ω	Largest Lyapunov exponent: Formula (3)	Largest Lyapunov exponent: Time series	Type of attractor
(1) Quasi-periodically forced Van-der-Pol's oscillator	5.0	–	–	5.0	0.006	$\sqrt{2} + 1.05$	−0.1213	0.0845	strange nonchaotic
	5.0	–	–	5.0	0.007	$\sqrt{2} + 1.05$	−0.2834	0.0684	strange nonchaotic
	5.0	–	–	5.0	0.003	$\sqrt{2} + 1.05$	0.1426	0.1468	strange chaotic
	5.0	–	–	5.0	0.002	$\sqrt{2} + 1.05$	0.1183	0.1232	strange chaotic
(2) Quasi-periodically forced pendulum	3.0	1.0	1.1	1.33	$0.5 - (\sqrt{5} - 1)/4$	$0.5 + (\sqrt{5} - 1)/4$	−0.0717	0.0104	strange nonchaotic
	0.5	1.0	1.1	0.80	$0.5 - (\sqrt{5} - 1)/4$	$0.5 + (\sqrt{5} - 1)/4$	0.0234	0.0282	strange chaotic
	0.005	0.027	0.2	0	$\sqrt{2}/10$	1.0	−0.0181	0.2030	strange nonchaotic
	0.010	0.027	0.2	0	$\sqrt{2}/10$	1.0	0.3801	0.3942	strange chaotic

while in the case of the strange chaotic attractor of Fig. 1(a) we estimated $d_I \cong 2$. The results of Table 2 show that the capacity dimension of the projection of the Poincaré map is near the prediction based on El Naschie's Peano dynamics namely $d_c \cong 2$.

By contrast the estimation for the information dimension is disappointing. The combination of having a projected Poincaré map and the use of a box counting method clearly has a strong negative effect on the accuracy of the calculation of the information dimension which may not be surprising since this is not a metric dimension.

4. PEANO-LIKE DYNAMICS AS A MODEL FOR STRANGE NONCHAOTIC BEHAVIOUR

Recently, El Naschie has made several suggestions regarding the use of space filling curves [19] as a carrier of fractal dynamics and proposed a prototype map for strange nonchaotic behaviour [18]. Although El Naschie's ideas seem to us to lack mathematical precision, they are quite bold and original and warrant further careful examination. In fact if they could be shown to be beyond any doubt correct then they may turn out to be extremely useful in understanding the nature of fully developed turbulence and strange nonchaotic behaviour.

We must also state from the outset that whether or not his method of deduction is correct, his conclusions do reinforce our own previous conclusions and this in turn may show that his method of deduction may be correct after all.

We would like to start by summarizing the main points of his theses as we understand it.

(a) Noting that an additional torsional movement could mimic the action of quasi-periodic forcing and the spiraling motion on a torus he introduced a 'S' form horseshoe-like map which he showed subsequently to lead to Peano-like space filling dynamics [18].

(b) Based on the ergodicity of the globally disjointed but locally space filling peano map (see Fig. 5) he concludes that the capacity dimension is $d_c = 2$ and the information dimension is $d_I = 1$.

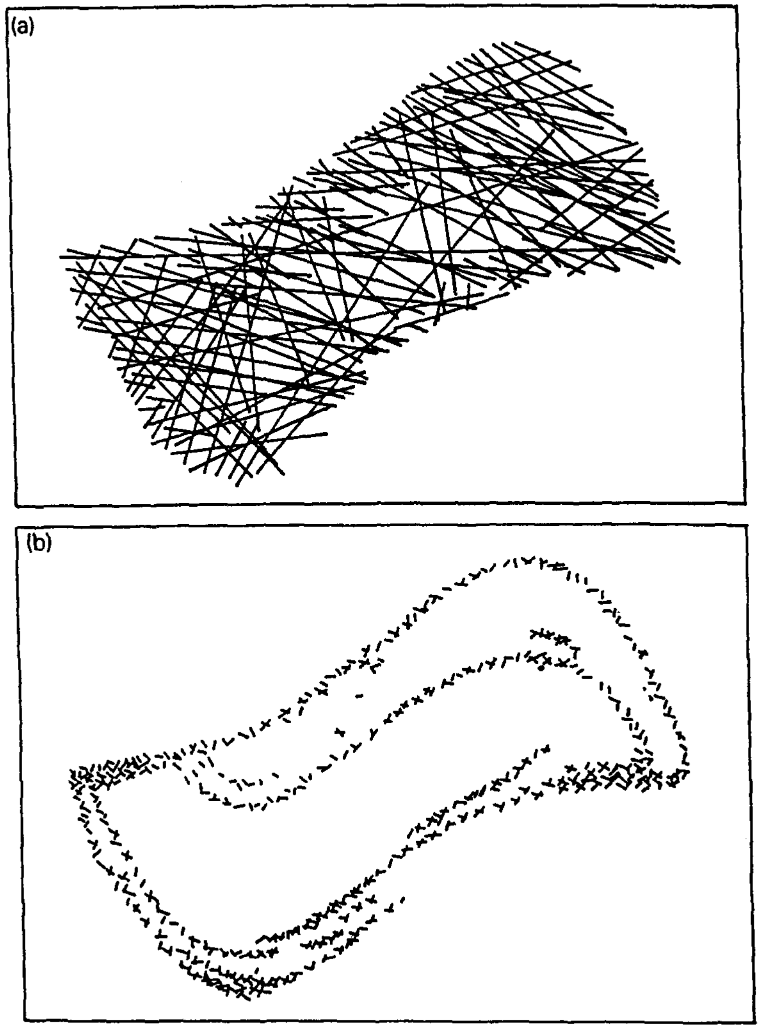

Fig. 3. The Poincaré maps of the two nearby trajectories (double Poincaré maps) for equation (1); (a) strange chaotic attractor of Fig. 1(a); (b) strange nonchaotic attractor of Fig. 1(b).

Table 2. Estimation of the capacity and information dimension of $x - \dot{x}$ projection of the Poincaré map for equation (1)

a	d	ω	Ω	Capacity dimension	Information dimension	Type of attractor
5.0	5.0	0.006	$\sqrt{2} + 1.05$	1.76	1.45	strange nonchaotic
5.0	5.0	0.007	$\sqrt{2} + 1.05$	1.82	1.52	strange nonchaotic
5.0	5.0	0.003	$\sqrt{2} + 1.05$	1.83	1.82	strange chaotic
5.0	5.0	0.002	$\sqrt{2} + 1.05$	1.96	1.93	strange chaotic

(c) He attributes the stabilization effect of quasi-periodic forcing to the shortening in the length of a horseshoe strip due to the torsional deformation.

(d) One of the conclusions which must draw attention is his claim that in four-dimensional phase space a strange set will typically have a Cantor-like fractal dimension $d_c \cong 4$. He

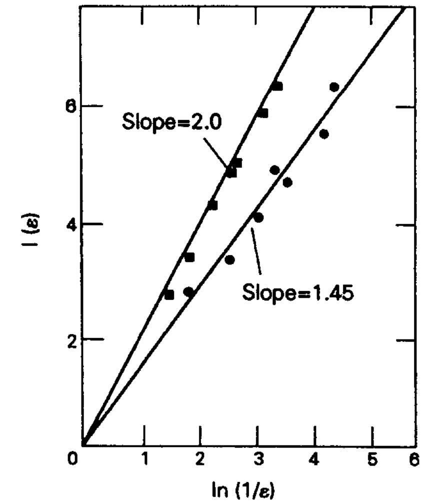

Fig. 4. Examples for the logarithmic plot used for determining dimension using box counting.

reaches this result using an unusual argument based on a scaling of dimensions. The rationale behind this unorthodox idea seems to be the following question. What is the Cantor set in two dimensions which corresponds to the one-dimensional middle third geometrical construction of a Cantor set. Such a set should be Cantorian with the same dimension regardless of the direction of the cross-section taken through the set. Putting it that way it is clear that such a set is not a Cartesian product of two perfect Cantor sets for which $d_c = \ln 4/\ln 3$. It is not even the Cantor target for which $d_c = 1 + \ln 2/\ln 3$. Based on the ratio between two unit areas corresponding to an Euclidian two-dimensional surface $A_E = (1)^2$ and that of an imaginary Cantor surface $A_c = (\ln 2/\ln 3)^2$ he found that this two-dimensional Cantor-like set is the Sierpinski gasket $d_c = \ln 3/\ln 2$. From there he proceeds to show that four-dimensional Euclidian manifolds are typically saturated by fractal dynamics corresponding to four-dimensional strange sets. After that the dimension of the strange set progressively exceeds the dimension of the hosting manifold by a substantial margin. He then takes what we feel to be quite a leap by arguing that four dimensions are therefore the critical line after which very strange phenomena may appear and that this is related to Newhouse and Ruelle-Takens turbulence scenario and Roessler's wrinkled attractors [20]. In the same spirit he proceeds to show that in five dimensions the fractal Cantor like set attractors have $d_c \cong 6.31$ and argues that five variables are what are needed to study fully developed turbulence. What interests us here however is the conclusion that if $d_c = 4$ is typical in four-dimensional phase space then for a two-frequency quasi-periodically forced oscillator we will have $d_c \cong 2$ on the surface of cross-section whether the strange set is chaotic or not. Consequently it is impossible to make a definite distinction between chaotic and nonchaotic attractors based on d_c alone. It is only when $d_c \cong 2$ but also $d_I \cong 1$ that we can infer the existence of a strange nonchaotic set [21]. This clearly agrees with our own previous conclusions.

CONCLUSIONS

The results reported here and in particular Table 2 shows that no distinction between strange chaotic and strange nonchaotic attractors can be made for quasi-periodically forced systems based on Lyapunov exponents estimated from a single time series only. This

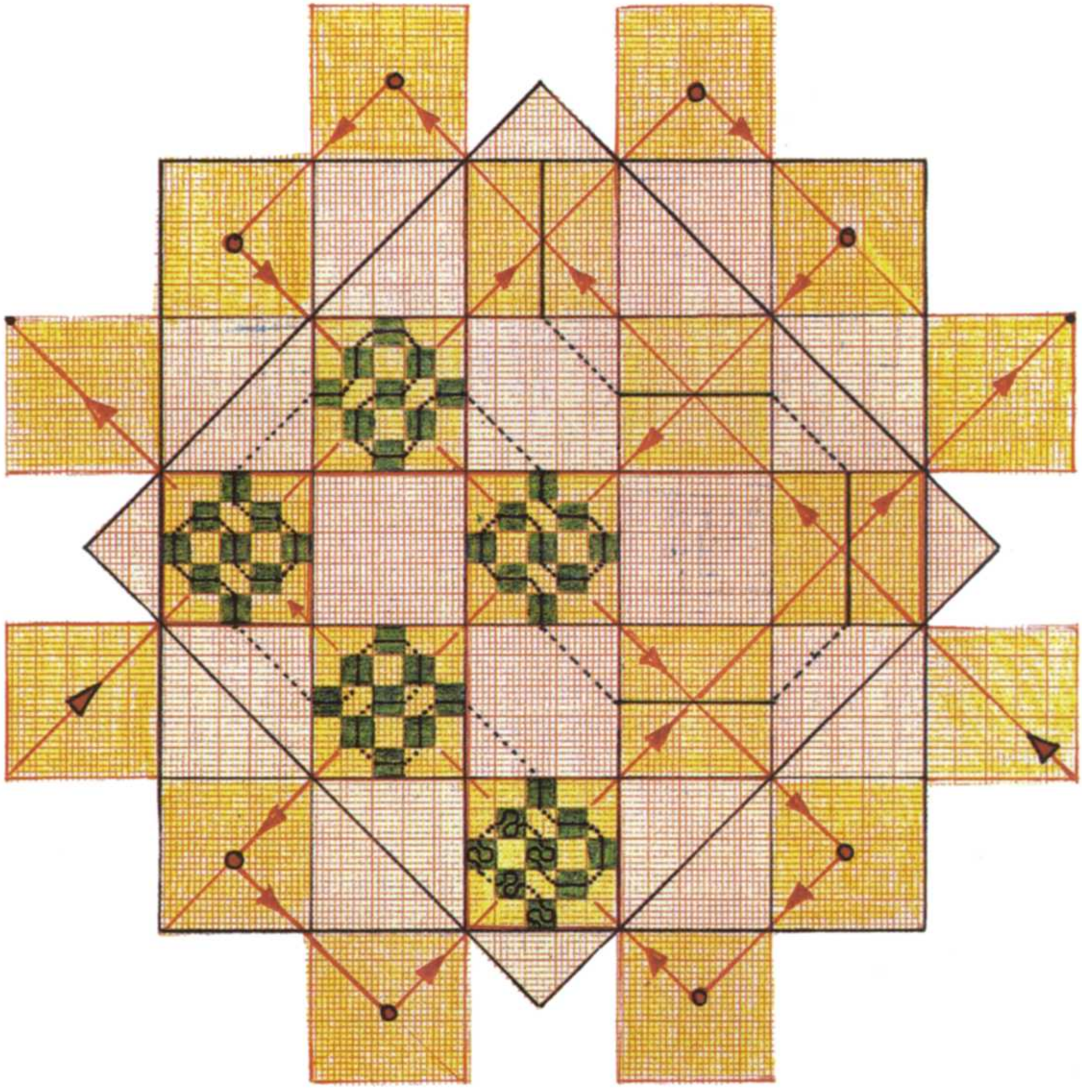

Fig. 5. Geometrical construction of a globally discrete and locally area filling peano curve which is oriented on a horseshoe-like discrete map. (Courtesy Prof. A. Hussein.)

procedure is viable only for strange chaotic attractors as can be seen from Table 2. The situation is of course different when the equations of the motion are differentiable. If the equations are known but are not differentiable then we can use the procedure outlined in Fig. 3 (a,b), where two solutions for nearby initial conditions are used. However, the experimentalist has usually only a single series of observations as is the case with atronomical data. In such cases we must study the data of one set of parameters for the usually unknown set of initial conditions. Information about sensitive dependence on initial conditions is then of no value.

The estimation of the information and the capacity dimension of an attractor on the Poincaré cross-section or its projections can provide good evidence for the existence of a strange nonchaotic attractor.

For the quasi-periodically forced system with strange nonchaotic attractor, considered here, the estimations for the map are $d_c \cong 2$ and $d_I \cong 1$ or at least $d_I \ll 2$. If we consider the dimension of the whole attractor then Kaplan–Yorke conjecture gives $d_I = 2$ while d_c must be larger than 3. According to El Naschie it is $d_c \cong 4$.

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