A note on randomness and strange behaviour

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Typical similarities and differences between strange chaotic and nonchaotic attractors in deterministic systems and random behaviour are discussed. It is shown that based on a single time series it is impossible to distinguish between these types of behaviour even using the technique of Lyapunov exponents.

In the last decade attention has been given to a class of dissipative dynamical systems that typically exhibit strange behaviour [1–3]. Such behaviour has been found in numerical experiments [3,4] as well as in experimental systems [5,6].

Recently two classes of strange attractors have been distinguished:

(a) A strange chaotic attractor – one which is geometrically "strange", i.e. the attractor is neither a finite set of points nor is it piecewise differentiable and one for which typical orbits have positive Lyapunov exponents.

(b) A strange nonchaotic attractor – one which is also geometrically "strange" but for which typical nearby orbits do not diverge exponentially with time [7–16].

Strange nonchaotic attractors have been found to be typical for quasiperiodically forced systems [7,13]. Recently they have been also observed in experimental systems [16]. Although one may doubt that these are periodic or quasiperiodic orbits with sufficiently long period, even in this case the period is longer than any reasonable observation and that is why their name is justified.

As both types of strange attractors look very similar (compare for example the Poincaré maps of figs. 1a and 1b), the value of the Lyapunov exponents seems to be the only quantity which allows us to distinguish these classes.

If we also consider a system forced by random noise we shall find that its behaviour is very similar to strange but deterministic behaviour. As an example consider the Poincaré maps of the system forced by random noise, fig. 2a, and by periodic force, fig. 2b.

In this paper we present some numerical experiments showing that it is impossible to distinguish between strange chaotic and nonchaotic deterministic behaviour and strange behaviour caused by random forcing.

For systems of which the equations of motion are explicitly known and the linearized equations exist there is a straightforward technique [7,8] for computing a complete Lyapunov spectrum.

For most of the experimental systems the equations of motion are not known [19] or are in the form for which the linearized equations do not exist [6,20]. In this case Lyapunov exponents are estimated based on the monitored long-term time series.

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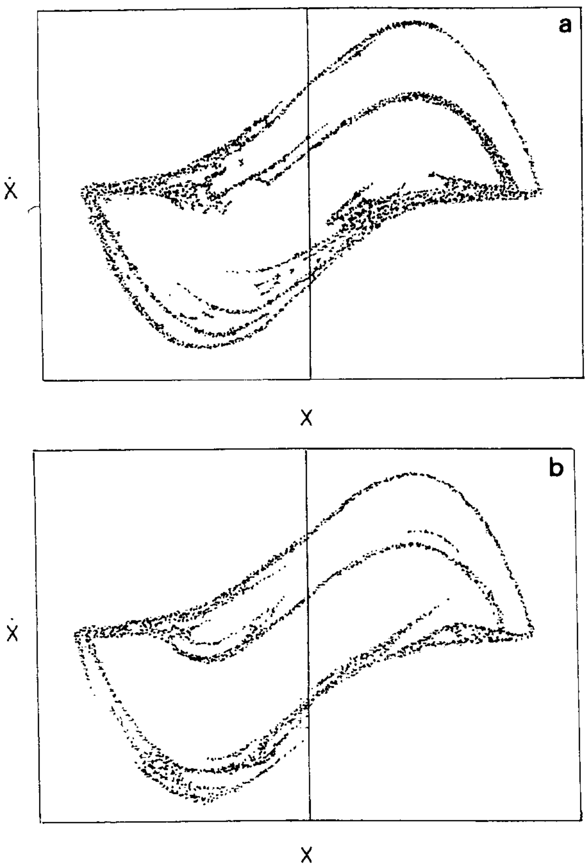

Fig. 1. Poincaré maps of the quasiperiodically forced van der Pol equation (1): $a=5.0, d=5.0, \Omega=\sqrt{2}+1.05$; (a) strange chaotic attractor $\omega=0.002$, the largest nonzero Lyapunov exponent $\lambda=0.1183$, (b) strange nonchaotic attractor $\omega=0.006$, $\lambda=-0.1213$.

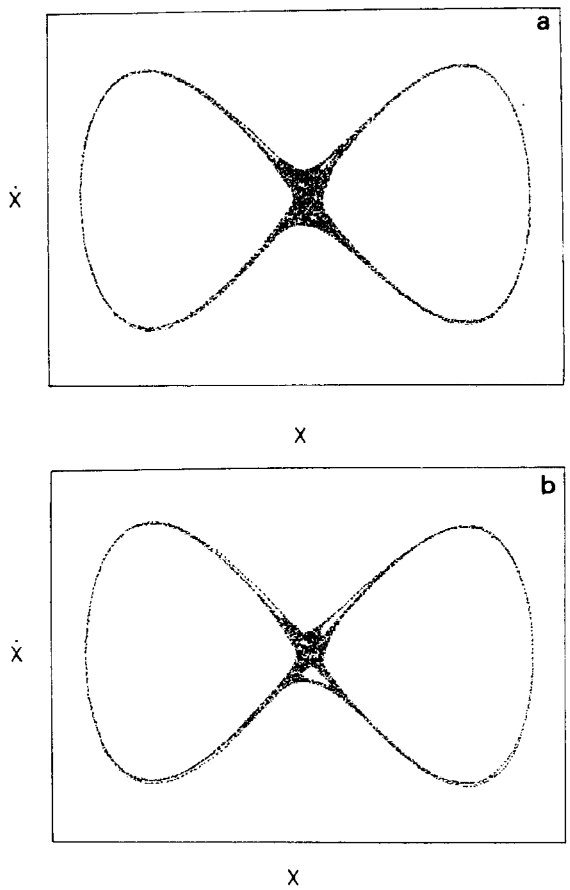

Fig. 2. Poincaré maps of the Duffing equations (3) and (4): $a=0.01$, $b=0.25$, $c=19/4\times18$; (a) random forcing by white noise with intensity $D=0.001$, (b) periodic forcing $d=0.001$, $\omega=1.0$.

First the attractor is reconstructed by the well-known technique with delay coordinates [21]. Our reconstructed attractor though defined by a single trajectory can provide points that may be considered to lie on different trajectories. It has been shown that in many cases this attractor has got a Lyapunov spectrum identical to that of the original attractor [17,22–25].

The technique of estimation of Lyapunov exponents based on the reconstructed attractor gives good results when we have at least one positive Lyapunov exponent in the spectrum [17].

In what follows we consider three systems:

(a) The quasiperiodically forced van der Pol equation,

$$\ddot{x}-a(1-x^2)\dot{x}+x=d\cos\omega t\cos\Omega t\,, \quad (1)$$

(b) the quasiperiodically forced pendulum,

$$\ddot{\Theta}+a\dot{\Theta}+b\sin\Theta=d+c(\cos\omega t+\cos\Omega t)\,, \quad (2)$$

(c) the Duffing equation forced by periodic force,

$$\ddot{x}+a\dot{x}-bx+cx^3=d\cos\omega t\,, \quad (3)$$

or random white noise $\eta(t)$ with intensity D,

$$\ddot{x}+a\dot{x}-bx+cx^3=\eta(t)\,. \quad (4)$$

For eqs. (1)–(3) linearized equations exist and we can compute Lyapunov exponents directly from the formula

$$\lambda = \lim_{t\to\infty} \frac{d(t)}{d(0)}, \tag{5}$$

where $d=\sqrt{y^2+\dot{y}^2}$, while y is a solution of a linearized equation.

In these cases we have computed Lyapunov exponents twice from the formula (5) and from time series based on the algorithm of Wolf et al. [17]. In numerical simulations the fourth-order Runge–Kutta method with time step $T/200$, where $T=2\pi/\omega$ has been used. Strange nonchaotic attractors have been observed up to $T=10^8$. The comparison of these results is shown in table 1.

From table 1 one finds that the calculation of the Lyapunov exponents from the explicitly known differentiable equation allows one to distinguish between strange chaotic and nonchaotic attractors. This distinction cannot be followed based on the Lyapunov exponents estimated from a single time series, as by this method we obtained positive values of Lyapunov exponents not only in the case of strange chaotic attractors but for strange nonchaotic attractors as well.

This result may look quite surprising but it is justified when we follow the method of estimation of Lyapunov exponents from the attractor reconstructed from a single time series. If a time series is irregular (not periodic, quasiperiodic) it is not distinctive from a chaotic one and the reconstructed attractor has got a complicated geometry. To estimate Lyapunov exponents from this attractor we have to obtain positive values for both strange chaotic and nonchaotic attractors, as the whole procedure explores the aperiodicity of time series and not the explicit dependence on initial conditions. As other methods of estimating Lyapunov exponents from time series [22–25] are also based on this method of attractor reconstruction it seems that using them similar results are very likely.

The difference between strange chaotic and nonchaotic attractors is visible when we plot a Poincaré map for two nearby trajectories and connect successive points of both trajectories (see fig. 3). In the case of strange nonchaotic behaviour it is visible that both trajectories stand close together, fig. 3a, while in the chaotic case a sensitive dependence on initial conditions is visible, fig. 3b.

There are some distinguishing properties between stochastic and chaotic systems. For example, the stochastic systems will not have a self-similar structure in the phase space, since stretching and folding of the manifolds do not occur. Indeed manifolds, as commonly defined, do not exist in the noisy case. Also, the transition to strange behaviour with changing the control parameter in the stochastically and periodically driven system are distinctly different. With periodic forcing the peaks in the power spectra accumulate in the period doubling cascade, as the control parameter is changing [26].

However, in many cases it is not possible to change

Table 1
The comparison of the values of Lyapunov exponents computed from formula (5) and estimated from time series.

Eq.	a	b	c	d	ω	Ω	D	Largest Lyapunov exponent		Type of attractor
								formula (5)	time series	
(1)	5.0	–	–	5.0	0.006	$\sqrt{2}+1.05$	–	−0.1213	0.0845	strange nonchaotic
	5.0	–	–	5.0	0.007	$\sqrt{2}+1.05$	–	−0.2834	0.0684	strange nonchaotic
	5.0	–	–	5.0	0.003	$\sqrt{2}+1.05$	–	0.1426	0.1468	strange chaotic
	5.0	–	–	5.0	0.002	$\sqrt{2}+1.05$	–	0.1183	0.1232	strange chaotic
(2)	3.0	1.0	1.1	1.33	$0.5-\frac{1}{4}(\sqrt{5}-1)$	$0.5+\frac{1}{4}(\sqrt{5}-1)$	–	−0.0717	0.0104	strange nonchaotic
	0.5	1.0	1.1	0.80	$0.5-\frac{1}{4}(\sqrt{5}-1)$	$0.5+\frac{1}{4}(\sqrt{5}-1)$	–	0.0234	0.0282	strange chaotic
	0.005	0.027	0.2	0	$\frac{1}{10}\sqrt{2}$	1.0	–	−0.0181	0.2030	strange nonchaotic
	0.010	0.027	0.2	0	$\frac{1}{10}\sqrt{2}$	1.0	–	0.3801	0.3942	strange chaotic
(3)	0.01	0.25	19/4×18	0.001	1.0	–	–	0.0864	0.0914	strange chaotic
(4)	0.01	0.25	19/4×18	–	–	–	0.001	–	0.0921	stochastically forced system

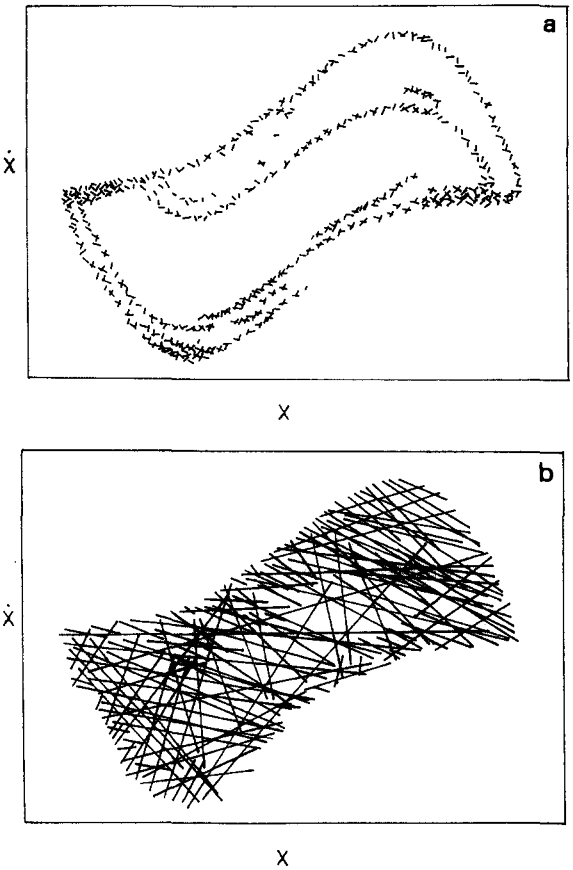

Fig. 3. Poincaré maps of two nearby trajectories of eq. (1); (a) strange chaotic attractor of fig. 1a, (b) strange nonchaotic attractor of fig. 1b.

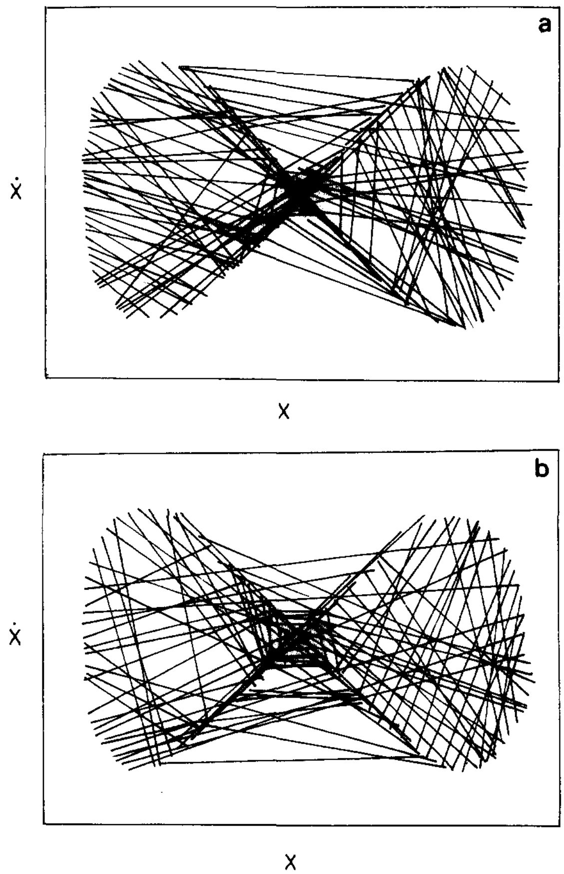

Fig. 4. Poincaré maps of two nearby trajectories of eqs. (3) and (4); (a) random forcing (as in fig. 2a), (b) periodic forcing (as in fig. 2b).

parameters in order to study the transition to strange behaviour. The experimentalist may have only a sequence of observations (a single time series). The typical examples are astronomical data. In this situation we will have to study only the data of one set of parameter values for one (usually unknown) set of initial conditions. Hence all information on various routes to chaos and sensitive dependence on initial conditions is of no value. In figs. 2a and 2b we show the Poincaré maps for chaotic and stochastically forced response of eqs. (3) and (4), while in figs. 4a and 4b we show the same maps for two nearby trajectories. In both cases one notices the sensitive dependence on initial conditions.

Generally, the stochastic system will possess a response that is less smooth than a deterministic one, which should be C^{∞} differentiable. Based on this property Sigeti and Horsthemke [27] found that the high frequency power spectrum is expected to distinguish between a deterministic and a stochastic system. They show that if a response is C^n differentiable (stochastic case) it will have a drop-off f^{-2n} in its power spectrum. This distinction cannot be applied to systems like those of figs. 2 and 4 where the noise is small enough and the response is still smooth enough to develop the above mentioned difference, as has been shown by Stone [28].

The analysis of the power spectra would distinguish between quasiperiodic orbits and strange nonchaotic attractors as has been shown in refs. [7,13,15], but distinction between strange nonchaotic and chaotic attractors is not straightforward.

The possibility of having a system possessing sensitivity to initial conditions caused by stochastic excitation as well as the possibility of having a system showing strange behaviour without sensitive dependence on initial conditions should not be overlooked. It seems that more care will have to be given in applying the procedure of estimation of Lyapunov exponents from time series or the result [27] to experimental data. The general conclusion that they imply can be misleading, as there are systems for which distinction between strange chaotic, strange nonchaotic and stochastic behaviour is impossible based on a single time series.

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