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Analytical conditions for strange chaotic and nonchaotic attractors of the quasiperiodically forced van der Pol equation

J. Brindley[a], T. Kapitaniak[a,1] and M.S. El Naschie[b]

[a]*Department of Applied Mathematical Studies and Center for Nonlinear Studies, University of Leeds, Leeds LS2 9JT, UK*
[b]*Sibley School of Mechanical and Aerospace Engineering, Cornell University, Ithaca, NY 14853-7501, USA*

We propose an analytical method for predicting the boundaries of the region in parameter space within which a strange nonchaotic attractor exists for a quasiperiodically forced van der Pol equation. One boundary is obtained from the condition for existence of a simple quasiperiodic solution to the equation. The other is shown to be well approximated by the condition for Hopf bifurcation in a suitably averaged system of equation. Good agreement with numerical calculations of Lyapunov exponents is demonstrated.

1. Introduction

Though for some time after the first recognition and description of a strange attractor the concepts of strange attractor and chaotic behaviour were widely regarded as synonymous, a number of authors (e.g. refs. [1–4] have more recently drawn attention to the existence of strange *nonchaotic* attractors (SNA), especially in quasiperiodically forced nonlinear oscillators. The existence of SNA's extends the range of possible routes to turbulence in systems in which they occur; many papers which address this feature are typified by refs. [5–8].

To avoid confusion, it is useful to define precisely what is meant by the various terms used. Thus "strange" describes the geometrical structure of the attractor; a strange attractor [4] is an attractor which is *not*:

- a finite set of points,
- a closed curve,
- a smooth or piecewise smooth surface,
- a volume bounded by a smooth or piecewise smooth surface.

The word "chaotic" on the other hand refers to the behaviour of trajectories on the attractor; the attractor is chaotic if nearby orbits diverge exponentially with time, implying infinite sensitivity to initial conditions. The divergence is measured by the Lyapunov exponent, and a strange nonchaotic attractor is an object whose geometry is none of the simple cases listed above, but for which typical orbits have nonpositive Lyapunov exponents.

In general, evidence of the existence of SNA's has arisen in numerical experiments using both iterative maps and differential equations. It has usually been presented in the form of spectra, or of (Poincaré) surfaces of section, supported by calculations of Lyapunov experiments. Romeiras and Ott [4] have gone further in arguing very convincingly that SMA's should occur commonly in quasiperiodically forced nonlinear systems of arbitrarily high dimensionality, provided that all orbits are attracted to a two-torus

[1]Permanent address: Institute of Applied Mechanics, Technical University of Lodz, Stefanowskiego 1/15, 90-924 Lodz, Poland.

on which the dynamics may be represented by an invertible map

$$\phi_{n+1} = g(\phi_n, \theta_n) \pmod{2\pi}, \quad \theta_{n+1} = \theta_n + 2\pi/\omega_2 \pmod{2\pi} \tag{1.1}$$

for certain classes of functions, g, representing the quasiperiodic forcing. (Here ω_2 is one of the two incommensurate frequencies of forcing, corresponding to $\omega - \Omega$ in (1.2) below.) Our own numerical integrations of a number of equations [7–9] support these ideas, but our objective in this paper is to present theoretical arguments which serve both to suggest reasons for the existence of SNA's over finite regions of parameter space, and also to give quantitative estimates of the boundaries of those regions. We consider a quasiperiodically forced van der Pol equation:

$$\begin{aligned} &\ddot{x} - 2\lambda(1 - \beta x^2)\dot{x} + \omega_0^2 x \\ &\quad = F \cos \omega t \cos \Omega t \\ &\quad = \tfrac{1}{2}F[\cos(\omega + \Omega)t + \cos(\omega - \Omega)t], \end{aligned} \tag{1.2}$$

where $\lambda < 1$, $\omega \ll \Omega$, $\omega \ll 1$, and ω, Ω are incommensurate.

First in section 2 we present some computed results, using as an example a system described fully by Qin et al. [10]. In section 3 we establish conditions for the existence of a quasiperiodic solution to eq. (1.2), and in section 4 we obtain conditions for a Hopf bifurcation in the averaged form of the equation. The results are discussed in section 4, where we argue that the two sets of conditions should bound region of existence of an SNA.

2. Numerical results

Eq. (1.2) is readily found to have substantial parameter ranges in which a nonchaotic strange attractor exists. The striking qualitative differences between the behaviour of trajectories near a nonchaotic attractor and those near a chaotic attractor are illustrated in figs. 1 and 2 in which the separation of points on two trajectories, starting nearby at $t = 0$, is illustrated at a hundred successive time iterates (10000–10100) for each of the two cases. Whilst in the chaotic case, the separation is large and rapidly varying (the iterates are in fact on quite different sheets of the attractor), in the nonchaotic case the separation is small: neighbouring trajectories stay close for long periods. The implications of this for the value of computations of trajectories whose initial state are subject to uncertainty or errors are clearly considerable.

The overall results may be typified by fig. 3, in which the regions of quasiperiodic, nonchaotic and chaotic response, deduced by calculation of Lyapunov exponents, are indicated. The results incorporate values of other parameters corresponding to the comprehensive investigation by Qin et al. [10] of the periodically forced van der Pol oscillator. Fig. 3 is to be compared with their figure 3. It appears that tongues of nonchaotic strange behaviour in our case have a roughly similar character to the tongues of periodic behaviour in the simply forced van der Pol oscillator which they have considered. It is possible of course that some of our nonchaotic regions in fact correspond to higher order quasiperiodic behaviour; this has not yet been fully checked.

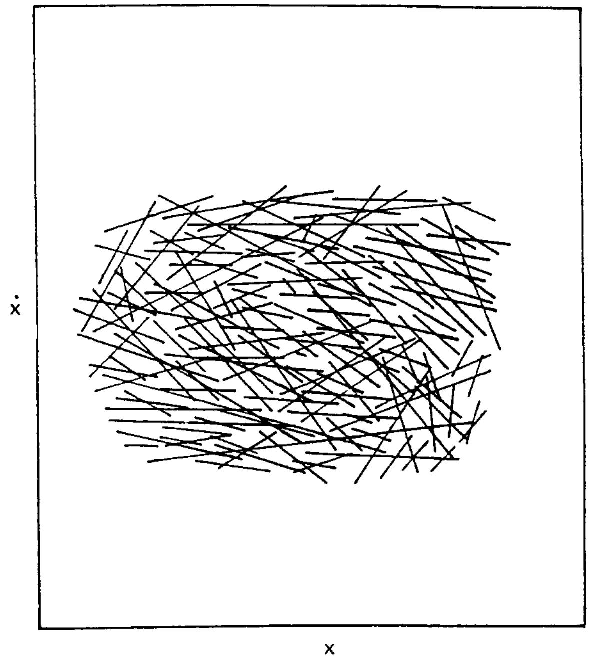

Fig. 1. Poincaré map for two nearby trajectories on the strange chaotic attractor (two successive points of each trajectory have been connected).

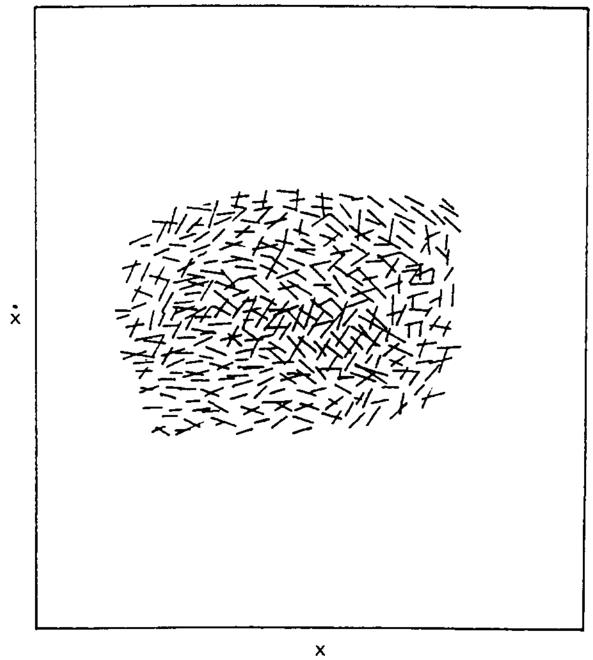

Fig. 2. Poincaré map for two nearby trajectories on strange nonchaotic attractor.

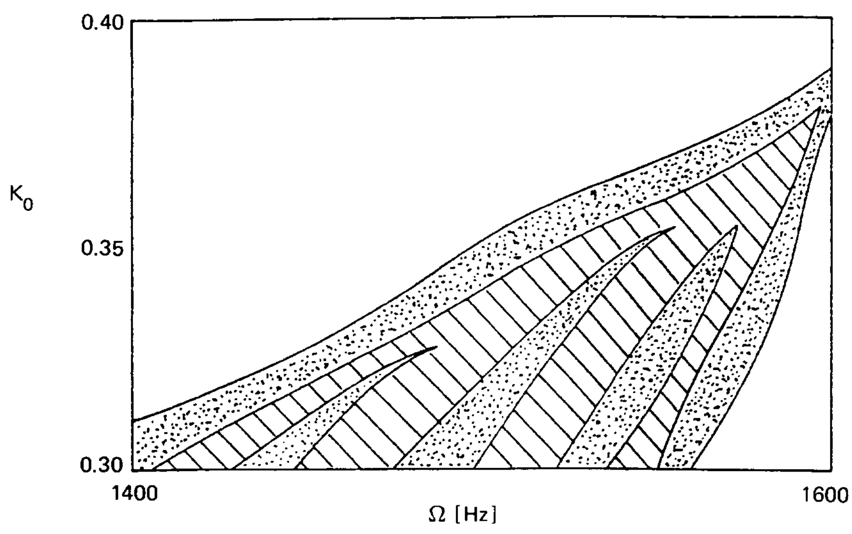

Fig. 3. Domains of strange chaotic and strange nonchaotic attractors of eq. (1.1); we have chosen values $\lambda = 0.3$, $\beta = \alpha K_1 RC/K_2$, $\omega_0^2 = 1/R^2C^2$, $F = K_0 U_m/R^3C^3$, $R = 10^5$, $C = 1.8 \times 10^{-9}$, $K_1 = 0.032$, $K_2 = 0.6$, $\omega = \sqrt{2}/10$, $\alpha = 0.0045$, $U_m = 10.9$ in order to compare with ref. [10].

3. Existence of quasiperiodic solutions

The simplest form of the solution of eq. (1.1) can be expressed as a two-frequency quasiperiodic function as follows:

$$x = a\cos[(\Omega - \omega)t + \phi_1] + b\cos[(\Omega + \omega)t + \phi_2]. \tag{3.1}$$

In this section we obtain an analytical condition determining the domain of existence of a solution of this kind.

First we recall some fundamental results regarding quasiperiodic operators.

Definition 1. A linear differential operator

$$Lz \equiv \frac{\mathrm{d}z}{\mathrm{d}t} - A(t)\,z \tag{3.2}$$

is said to be quasiperiodic if $A(t)$ is a quasiperiodic matrix.

Definition 2. A quasiperiodic operator L is said to be regular if for any quasiperiodic function $f(t)$ the equation

$$Lz = f(t) \tag{3.3}$$

has at least one solution bounded for all $t \in \mathbb{R}$.

Now let $\phi(t)$ be the fundamental matrix of the linear homogeneous equation

$$Lz = 0 \tag{3.4}$$

satisfying the initial condition $\phi(0) = I$, where I is the unit matrix. Then we have

Proposition 1 [11]. L is regular if and only if there is a square matrix P such that

(a) $P^2 = P$,

(b) $\|\phi(t)\,P\phi^{-1}(s)\| \le C\,\mathrm{e}^{-\sigma(t-s)}$ for $t \ge s$,

(c) $\|\phi(t)\,(I-P)\phi^{-1}(s)\| \le C\,\mathrm{e}^{-\sigma(s-t)}$ for $t < s$,

where C and σ are positive numbers.

Proposition 2 [12]. If a quasiperiodic operator L with periods $\omega_1, \omega_2, \ldots, \omega_m$ defined by (3.2) is regular, then for any quasiperiodic function $f(t)$ with periods $\omega_1, \omega_2, \ldots, \omega_m$ the differential equation (3.3) possesses a unique quasiperiodic solution $z = z(t)$ with the same periods given by

$$z(t) = \int_{-\infty}^{\infty} G(t,s)\,f(s)\,\mathrm{d}s, \tag{3.5}$$

where

$$G(t,s) = \phi(t)\,P\phi^{-1}(s) \qquad \text{for } t \geq s,$$
$$\quad = \phi(t)(I-P)\phi^{-1}(s) \quad \text{for } t < s. \tag{3.6}$$

$G(t,s)$ is the Green function for L, and satisfies the inequality

$$\|G(t,s)\| \leq C\,\mathrm{e}^{-\sigma}(t-s). \tag{3.7}$$

The condition for existence of the quasiperiodic solution (3.1) to eq. (1.1) is now based on the following theorem.

Theorem 1 [12]. Consider a nonlinear differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = X(t,z), \tag{3.8}$$

where z and $X(t,z)$ are vectors and $X(t,z)$ is quasiperiodic in t with periods $\omega_1, \omega_2, \ldots, \omega_m$ and is continuously differentiable with respect to z within some region D of the z-space.

Suppose that there is a quasiperiodic function $z_0(t)$ with periods $\omega_1, \omega_2, \ldots, \omega_m$ such that

$$z_0(t) \in \mathrm{D}, \quad \left\| \frac{\mathrm{d}z_0}{\mathrm{d}t}(t) - X[t, z_0(t)] \right\| \leq r$$

for all $t \in \mathbb{R}$. Further suppose that there exists a positive number δ, a non-negative number $K < 1$ and a quasiperiodic matrix $A(t)$ with periods $\omega_1, \omega_2, \ldots, \omega_m$ such that

(a) the quasiperiodic differential operator L defined by (3.2) is regular,

(b)

$$\mathrm{D}_\delta = \{z, \|z - z_0(t)\| \leq \delta \text{ for some } t \in \mathbb{R}\} \subset \mathrm{D},$$

$$\|\psi(t,z) - A(t)\| \leq \frac{K}{M} \text{ whenever } \|z - z_0(t)\| \leq \delta,$$

$$\frac{Mr}{1-K} \leq \delta.$$

Here $\psi(t,z)$ is the Jacobian matrix of $X(t,z)$ with respect to z and

$$M = 2C/\sigma$$

where C and σ are positive numbers satisfying (3.7).

Then the given equation (3.8) possesses a solution $z = \hat{z}(t)$ quasiperiodic in t with periods $\omega_1, \omega_2, \ldots, \omega_m$ such that

$$\|z_0(t) - \hat{z}(t)\| \leq \frac{Mr}{1-K} \tag{3.9}$$

for all $t \in \mathbb{R}$.

If we now rewrite our equation (1.1) as

$$\frac{\mathrm{d}z}{\mathrm{d}t} = A(\lambda)\, z + \phi(t) + \lambda\eta(z), \tag{3.10}$$

where

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A(\lambda) = \begin{pmatrix} 0 & 1 \\ \omega_0^2 & 2\lambda \end{pmatrix}$$

$$\eta(z) = \begin{pmatrix} 0 \\ -2\beta x^2 y \end{pmatrix}, \quad \phi(t) = \begin{pmatrix} 0 \\ \frac{1}{2}F[\cos(\Omega-\omega)t + \cos(\Omega+\omega)t] \end{pmatrix},$$

we can use theorem 1 to obtain the following result:

If $0 < \lambda < 1$ and if the constant

$$C = \max\left[\frac{|F|}{2}\left(\frac{1}{|\omega_0^2 - (\Omega-\omega)^2|} + \frac{1}{|\omega_0^2 - (\Omega+\omega)^2|}\right), \frac{|F|}{2}\left(\frac{\Omega-\omega}{|\omega_0^2 - (\Omega-\omega)^2|} + \frac{\Omega+\omega}{|\omega_0^2 - (\Omega+\omega)^2|}\right)\right]$$

fulfills the inequality

$$C \le \sqrt{\sqrt{\omega_0^2 - \lambda^2}\,/52\beta\sqrt{2+2\lambda}}\,, \tag{3.11}$$

then eq. (1.1) possesses a quasiperiodic solution (2.1).

We establish this result as follows. Consider a linear differential operator depending on λ such that

$$L(\lambda)\, w = \frac{\mathrm{d}w}{\mathrm{d}t} - A(\lambda)\, w, \tag{3.12}$$

where $L(\lambda)$ is regular as a quasiperiodic operator for $\lambda < 1$ and the Green function for $L(\lambda)$ is given by

$$G(t,s) = 0, \quad t \ge s,$$

$$G(t,s) = -\mathrm{e}^{\lambda(t-s)} \begin{pmatrix} \cos\mu(t-s) - \frac{\lambda}{\mu}\sin\mu(t-s) & \frac{1}{\mu}\sin\mu(t-s) \\ -\frac{1}{\mu}\sin\mu(t-s) & \cos\mu(t-s) + \frac{\lambda}{\mu}\sin\mu(t-s) \end{pmatrix}, \quad t < s, \tag{3.13}$$

where $\mu = \sqrt{\omega_0^2 - \lambda^2}$.

Then we have

$$\|G(t,s)\| \le \frac{\sqrt{2+2\lambda}}{\mu}\,\mathrm{e}^{-\lambda|t-s|} \tag{3.14}$$

and the quasiperiodic solution to the linear equation

$$L(\lambda)\, z = \phi(t)$$

is

$$z + z_0(t, \lambda) = \begin{pmatrix} x_0(t, \lambda) \\ y_0(t, \lambda) \end{pmatrix}, \tag{3.15}$$

where

$$\begin{aligned} x_0(t, \lambda) &= \frac{\frac{1}{2}F}{\left(\omega_0^2 - \nu_1^2\right)^2 + 4\lambda^2\nu_1^2}\left[\left(\omega_0^2 - \nu_1^2\right)\cos\nu_1 t - 2\lambda\nu_1 \sin\nu_1 t\right] \\ &\quad + \frac{\frac{1}{2}F}{\left(\omega_0^2 - \nu_2\right)^2 + 4\lambda^2\nu_2^2}\left[\left(\omega_0^2 - \nu_2^2\right)\cos\nu_2 t - 2\lambda\nu_2 \sin\nu_2 t\right], \\ y_0(t, \lambda) &= \frac{\frac{1}{2}F\nu_1}{\left(\omega_0^2 - \nu_1^2\right)^2 + 4\lambda^2\nu_1^2}\left[-\left(\omega_0^2 - \nu_1^2\right)\sin\nu_1 t - 2\lambda\nu_1 \cos\nu_1 t\right] \\ &\quad + \frac{\frac{1}{2}F\nu_2}{\left(\omega_0^2 - \nu_2^2\right)^2 + 4\lambda^2\nu_2^2}\left[-\left(\omega^2 - \nu_2^2\right)\sin\nu_2 t - 2\lambda\nu_2 \cos\nu_2 t\right] \end{aligned} \tag{3.16}$$

and we have written $\nu_1 = \Omega - \omega$, $\nu_2 = \Omega + \omega$.

Now let

$$C = \max\left(\frac{\frac{1}{2}F}{|\omega_0^2 - \nu_1^2|} + \frac{\frac{1}{2}F}{|\omega_0^2 - \nu_2^2|}, \frac{\frac{1}{2}F\nu_1}{|\omega_0^2 - \nu_1^2|} + \frac{\frac{1}{2}F\nu_2}{|\omega_0^2 - \nu_2^2|}\right), \tag{3.17}$$

so that we have

$$|x_0(t, \lambda)| \le C, \quad |y_0(t, \lambda)| \le C \tag{3.18}$$

for all $t \in \mathbb{R}$ and $0 < \lambda < 1$.

Then using the inequalities (3.18), we can estimate the residual function for $z_0(t, \lambda)$ as follows:

$$\begin{aligned} &\left\| \frac{\mathrm{d}z_0}{\mathrm{d}t}(t, \lambda) - A(\lambda)\, z_0(t, \lambda) - \phi(t) - \lambda\eta(z_0(t, \lambda)) \right\| \\ &\quad = \|-\lambda\eta(z_0(t, \lambda))\| \le 2\lambda\beta |x_0^2(t, \lambda)\, y_0(t, \lambda)| \le 2\lambda\beta C^3. \end{aligned} \tag{3.19}$$

Thus, in the notation of theorem 1, we can choose

$$r = 2\lambda\beta C^3 \tag{3.20}$$

and let

$$\mathrm{D}_C = \{z, \|z\| \le 2C\}, \quad \mathrm{D}' = \bigcup_{T \in \mathbb{R}} \{z, \|z - z_0(t, \lambda)\| \le C\}, \tag{3.21}$$

so that

$$z_0(t,\lambda) \in \mathrm{D}_C \quad \text{for only } t \in \mathbb{R} \quad \text{and} \quad \mathrm{D}' \subset \mathrm{D}_C.$$

Finally, if $\psi(z,\lambda) =$ Jacobian matrix of the right-hand side of (3.12) then

$$\|\psi(z,\lambda) - A(\lambda)\| \le 2\lambda\beta(2|y| + |x|), \quad |x| \le 24\lambda\beta C^2 \tag{3.22}$$

and so

$$M = 2\sqrt{2+2\lambda}\Big/\lambda\sqrt{\omega_0^2 - \lambda^2}\,. \tag{3.23}$$

and it follows that, if (3.10) holds, the results of theorem 1 (eq. (3.9)) guarantee the existence of a quasi-periodic solution of the form (3.1).

4. Hopf bifurcation in the averaged system

In this section we consider the bifurcation behaviour of a simpler system of ODEs obtained from (1.1) by a suitable averaging procedure. We use a multifrequency averaging technique (over two periods $T_1 = 2\pi/\omega$ and $T_2 = 2\pi/\Omega$) we exploit the fact that $\omega \ll \Omega$ and $\omega \ll 1$ and assume the solution of the equation (1.1) may be written in the following form:

$$x \approx a(t)\cos\omega t\cos\Omega t + b(t)\cos\omega t\sin\Omega t, \tag{4.1}$$

where $a(t)$ and $b(t)$ are slowly varying amplitudes.

Expression (4.1) is less general than solution (3.1), but in the case of (3.1) it would be difficult to compute $a(t)$ and $b(t)$.

The first and second derivatives of x are taken to be

$$\dot{x} \approx -\Omega a(t)\cos\omega t\sin\Omega t + \Omega b(t)\cos\omega t\cos\Omega t,$$

$$\ddot{x} \approx -\Omega^2 a(t)\cos\omega t\cos\Omega t - \Omega^2 b(t)\cos\omega t\sin\Omega t - \dot{a}(t)\,\Omega\cos\omega t\sin\Omega t + \dot{b}(t)\,\Omega\cos\omega t\cos\Omega t. \tag{4.2}$$

In the derivation of (4.2) besides the usual assumption for transformation from the fast variables $\dot{x}, \ddot{x}$ to slow variables a, b, i.e. that

$$a(t)\cos\Omega t + b(t)\sin\Omega t = 0,$$

we have also assumed that

$$-\omega\Omega[a(t)\sin\omega t\cos\Omega t - b(t)\sin\omega t\sin\Omega t] \approx 0,$$

since $\omega \ll 1$.

Substituting (4.1) and (4.2) into (1.1) and averaging over T_1 and T_2, we obtain the following equations:

$$\dot{a} = -\bar{\omega}b - \lambda a - \tfrac{1}{2}\lambda\beta a(a^2+b^2), \quad \dot{b} = \bar{\omega}a - \lambda a - \tfrac{1}{8}\lambda\beta b(b^2+a^2) - \tfrac{1}{2}F, \tag{4.3}$$

where $\bar{\omega} = (\Omega^2-1)/2\Omega$.

Linearization around the fixed point (a_0, b_0) leads to

$$\dot{a}' = \lambda\left[1-\beta\left(\tfrac{3}{2}a_0^2 - \tfrac{1}{2}b_0^2\right)\right]a' + \left(-\bar{\omega} - \lambda\beta a_0 b_0\right)b', \quad \dot{b}' = \left(\bar{\omega} - \tfrac{1}{4}\lambda\beta a_0 b_0\right)a' + \lambda\left[1-\beta\left(\tfrac{3}{8}b_0^2 - \tfrac{1}{8}a_0^2\right)\right]b', \tag{4.4}$$

where

$$a = a_0 + a', \quad b = b_0 + b'.$$

Hopf bifurcation points can be found by considering the roots of the characteristic equation

$$\delta^2 - (A+D)\delta + AD - BC = 0,$$

where

$$A = \lambda\left[1-\beta\left(\tfrac{3}{2}a_0^2 - \tfrac{1}{2}b_0^2\right)\right], \quad B = \left(-\bar{\omega} - \lambda\beta a_0 b_0\right),$$
$$C = \left(\bar{\omega} - \tfrac{1}{4}\lambda\beta a_0 b_0\right), \quad D = \lambda\left[1-\beta\left(\tfrac{3}{8}b_0^2 - \tfrac{1}{8}a_0^2\right)\right].$$

5. Discussion and conclusions

The essential results of this paper are all summarised in fig. 4. Superimposed on the results of calculations of Lyapunov exponents for the quasiperiodically forced van der Pol equation (1.1) are results obtained by semi-analytical methods in sections 3 and 4. The dashed line bounds the region within which quasiperiodic solutions

$$x = A\cos\omega t\cos\Omega t$$

can exist. It shows good agreement with the boundary of strange nonchaotic behaviour in the numerical results, as might be expected.

The agreement between the curve of Hopf bifurcation in the averaged equations and the boundary dividing nonchaotic from chaotic behaviour is less obviously expected. At this stage we can do little other than report it and note that the agreement is "robust" in the sense that, e.g. when λ is allowed to vary from its value, 0.3, used in fig. 4, we find that the region of chaotic behaviour in the λ-direction, at a fixed value of K_0, Ω, is well predicted by the Hopf bifurcation in the averaged system (fig. 5). Noting the tongue-like regions of subharmonic response computed by Qin et al. [10], it is interesting to conjecture that the tongues of nonchaotic behaviour embedded in the main chaotic region of fig. 4 are associated

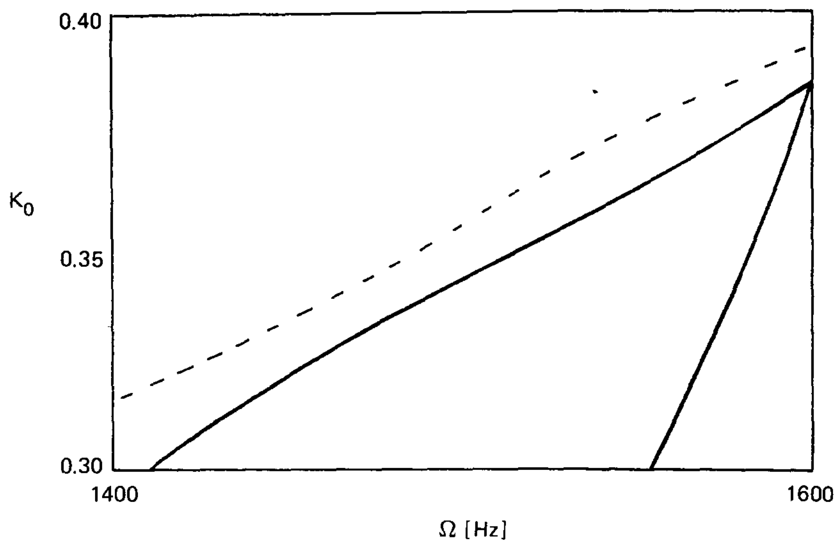

Fig. 4. Boundary of existence of the solution (3.1) (broken line) and boundaries of Hopf bifurcation (solid line) for eq. (1.1) (system parameters as in fig. 3).

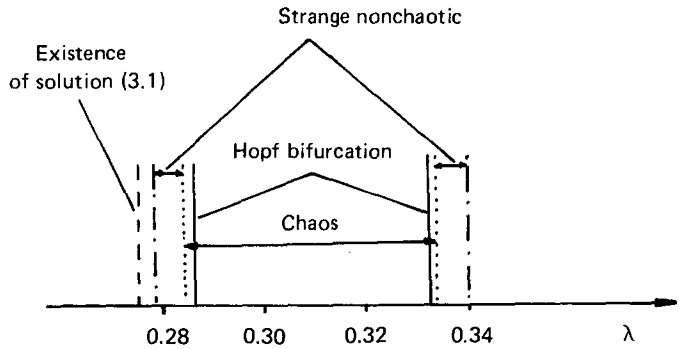

Fig. 5. Domains of strange chaotic and strange nonchaotic attractors together with the boundary of existence of the solution (2.1) and Hopf bifurcation: $K_0 = 0.33$, $\Omega = 1500$.

with boundaries of regions of "subharmonic" quasi-periodic solutions of the form

$$x = \sum_{n=1}^{N} A_n \cos \omega t \cos \frac{\omega t}{n}.$$

Finally we remark that similar nonchaotic strange attractors are found in forced Duffing type equations [9].

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