Analytic predictors for strange non-chaotic attractors

J. Brindley and T. Kapitaniak [1]

Department of Applied Mathematical Studies and Center for Nonlinear Studies, University of Leeds, Leeds LS2 9JT, UK

Received 29 August 1990; revised manuscript received 18 January 1991; accepted for publication 4 March 1991
Communicated by A.P. Fordy

Analytic conditions are used to predict the bounds in parameter space of the regions of existence of a non-chaotic strange attractor for the quasi-periodically forced van der Pol equation. One bound arises from the condition for existence of a simple quasi-periodic response to the forcing; the other appears to be related to the occurrence of a Hopf bifurcation in the averaged form of the equation.

1. Introduction

Chaotic behaviour of solutions to a system of differential equations or maps is characterised by the exponential divergence of neighbouring trajectories near to or on an attractor, and by the "strangeness" of the attractor itself. By strangeness we mean having a geometrical configuration not definable by a finite number of points, smooth curves or surfaces.

It is now well established however [1–8] that some systems are characterised by non-chaotic strange attractors (NSAs). In this case the attractor is geometrically strange, but neighbouring trajectories do not diverge exponentially. There is no positive Lyapunov exponent through there may be one or more which take the value zero. They are clearly distinguishable from quasi-periodicity by the form of the spectrum [4].

NSAs seem to form part of the normal pattern of behaviour in quasi-periodically forced nonlinear oscillators [4,7,8], and their presence has been demonstrated by a number of numerical investigations. In this note we present two analytic approaches which give good approximations for the boundaries in parameter space of regions of existence of non-chaotic attractors. One approach seeks rigorous bounds on the region of existence of a simple quasi-periodic solution; the other examines the condition for Hopf bifurcation in an averaged form of the system. Results of both approaches are demonstrated for the case of a quasi-periodically forced van der Pol equation, for which the regions of existence in parameter space of NSAs have been deduced from numerical calculations.

[1] Permanent address: Institute of Applied Mechanics, Technical University of Lodz, Stefanowskiego 1/15, 90-924 Lodz, Poland.

2. Non-chaotic strange attractor for the quasi-periodically forced van der Pol equation

Numerical evidence of the existence of a non-chaotic strange attractor for the quasi-periodically forced van der Pol equation

$$\ddot{x}-2\lambda(1-\beta x^2)\dot{x}+\omega_0^2 x=F\cos\omega t\cos\Omega t$$
$$=\tfrac{1}{2}F\{\cos[(\omega-\Omega)t]+\cos[(\omega+\Omega)t]\} \qquad (2.1)$$

has been presented elsewhere [7–9].

We can summarise typical results in figs. 1 and 2. The parameter values in the figures have been chosen to correspond to those used by Qin et al. [10] in their extensive review of behaviour in a van der Pol oscillator forced at a single frequency. Note that the tongues of non-chaotic behaviour in fig. 1 bear striking resemblance to the tongues of subharmonic response in their investigation.

In our numerical investigations we used the ODE

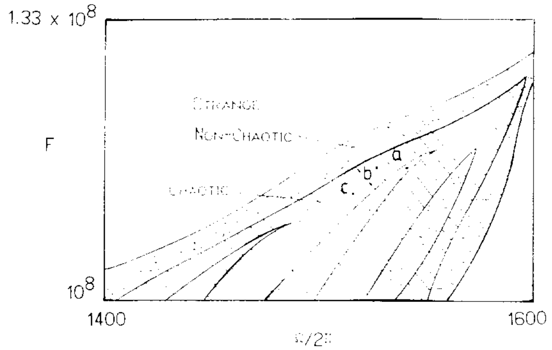

Fig. 1. Domains of strange chaotic (hatched) and strange non-chaotic (dotted) attractors of eq. (2.1); we have chosen values $\lambda=3\times10^3$, $\beta=2.4\times10^{-3}$, $\omega_0^2=5500$, $\omega=\sqrt{2}/10$, in order to compare with ref. [10]. Note that $F=10^8$ corresponds to $K_0=0.3$ and that $\Omega/2\pi=f$.

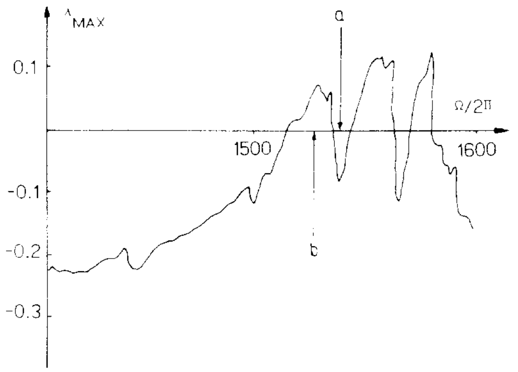

Fig. 2. The largest Lyapunov exponent λ_{max} versus Ω: $F=1.15\times10^8$.

procedure of Wolf et al. [14] to compute Lyapunov exponents. Eq. (2.1) and the appropriate linearised equation have been solved by the fourth-order Runge–Kutta method with integration step $\pi/100(\Omega+\omega)$.

We should stress that the actual separation of pairs of trajectories originating from nearby starting conditions does remain small in these regions of non-chaotic behaviour, even though the attractor near to which each trajectory remains is indeed strange in a geometrical sense. This is exemplified in fig. 3, where we show the variation of separation distance for a sequence of time steps in the integration of eq. (2.1), and, for contrast, a sequence of separations for a similar pair of trajectories for parameter values corresponding to a chaotic attractor. Each sequence relates to the same block, steps 10000–11000, of the computation. This difference is obviously of great practical value in problems in which there is uncertainty in initial data. Plots such as fig. 3 to describe characteristic properties of NSAs and chaotic ones have to be long enough as one can find short intervals of time where separation vectors stay approximately the same even in chaotic systems with large Lyapunov exponents.

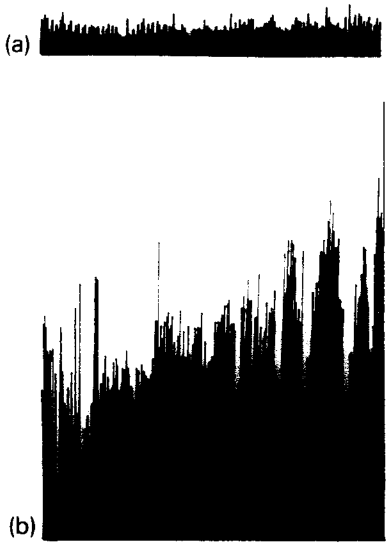

Fig. 3. The separation of nearby trajectories of a strange non-chaotic attractor (a) at point a in figs. 1 and 2 and a chaotic attractor (b) at point b in figs. 1 and 2.

3. Analytical prediction of existence of non-chaotic strange attractors

The results of fig. 1, and of most other studies of non-chaotic strange attractors, have been obtained by direct numerical calculation. Here we exploit results of a theorem by Urabe [12] to establish a condition for existence of a simple quasi-periodic solution,

$$x=a\cos[(\Omega-\omega)t+\phi_1]+b\cos[(\Omega+\omega)t+\phi_2]\,, \tag{3.1}$$

to eq. (2.1).

Briefly, the condition may be stated: if the constant C, given by

$$C=\max\left\{\frac{|F|}{2}\left(\frac{1}{|\omega_0^2+(\Omega-\omega)^2|}+\frac{1}{|\omega_0^2-(\Omega+\omega)^2|}\right),\ \frac{|F|}{2}\left(\frac{\Omega-\omega}{|\omega_0^2-(\Omega-\omega)^2|}+\frac{\Omega+\omega}{|\omega_0^2-(\Omega+\omega)^2|}\right)\right\}, \tag{3.2}$$

satisfies the inequality

$$C\leqslant\frac{1}{(52\xi)^{1/2}}\left(\frac{\omega_0^2-\lambda^2}{2+2\lambda}\right)^{1/4}, \tag{3.3}$$

then eq. (2.1) has a quasi-periodic solution of the form (3.1). Further details of the modified theorem are included in ref. [13]. This condition is readily applied to eq. (2.1), and, choosing $\lambda=3\times10^3$, $\beta=2.4\times10^{-3}$, $\omega_0^2=5500$, $\omega=\sqrt{2}/10$ we find the boundary indicated in fig. 4 for the region of existence of quasi-periodic solutions. Beyond this boundary quasi-periodic solutions no longer exist, but the Lyapunov exponent is still negative and hence nearby trajectories do not in general diverge exponentially. Extensive computations of attractors [8] for eq. (2.1) show that this behaviour is "normal", and the whole sequence seems to fit the concept [4] that both quasi-periodic and non-chaotic strange attractors lie on a particular three-torus embedded in the full four-dimensional space. Only when this torus is destroyed is chaos observed.

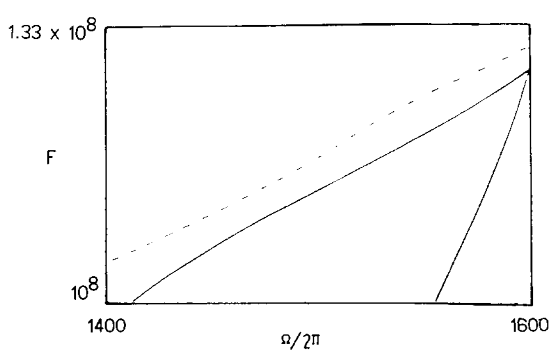

Fig. 4. Boundary of existence of the solution (3.1) (dashed line) and boundaries of Hopf bifurcation (solid line) for eq. (2.1) (system parameters as in fig. 1).

4. An averaging approach

An alternative approach to establishing boundaries of quasi-period response is through averaging. We use a multifrequency averaging technique (over two periods $T_1=2\pi/\omega$ and $T_2=2\pi/\Omega$) and exploit the fact that $\omega\ll\Omega$ and $\omega\ll1$, assuming that the solution of (2.1) may be written in the form

$$x\approx a(t)\cos\omega t\cos\Omega t+b(t)\cos\omega t\sin\Omega t, \tag{4.1}$$

where $a(t)$ and $b(t)$ are slowly varying amplitudes. Expression (4.1) is less general than solution (3.1), but in the case of (3.1) it would be difficult to compute $a(t)$ and $b(t)$. The first and second derivatives of x are taken to be

$$\dot{x}\approx-\Omega a(t)\cos\omega t\sin\Omega t+\Omega b(t)\cos\omega t\cos\Omega t,$$

$$\ddot{x}\approx-\Omega^2a(t)\cos\omega t\cos\Omega t-\Omega^2b(t)\cos\omega t\sin\Omega t-a(t)\Omega\cos\omega t\sin\Omega t+b(t)\Omega\cos\omega t\cos\Omega t. \tag{4.2}$$

In the derivation of (4.2) besides the usual assumption for transformation from fast variables $\dot{x}$, $\ddot{x}$ to slow variables a, b, i.e. that

$$a(t)\cos\Omega t+b(t)\sin\Omega t=0,$$

we have also assumed that

$$-\omega\Omega[a(t)\sin\omega t\cos\Omega t-b(t)\sin\omega t\sin\Omega t]\approx0,$$

since $\omega\ll1$.

After averaging over T_1 and T_2 and linearisation around the fixed point (a_0, b_0) of the averaged equations, Hopf bifurcation points can be found by considering the roots of the characteristic equation

$$\delta^2-(A+D)\delta+AD-BC=0,$$

where

$$A=\lambda[1-\beta(\tfrac{3}{2}a_0^2-\tfrac{1}{2}b_0^2)],\quad B=-\omega-\lambda\beta a_0b_0,$$

$$C=\omega-\tfrac{1}{4}\lambda\beta a_0b_0,\quad D=\lambda[1-\beta(\tfrac{3}{8}b_0^2-\tfrac{1}{8}a_0^2)],$$

$$\bar{\omega}=(\Omega^2-1)/2\Omega.$$

5. Results and discussion

We present the results of sections 4 and 5 in fig. 4, in which the dashed line bounds the region of existence of quasi-periodic solutions. Comparison with fig. 1 confirms that it gives a good prediction of the boundary of regions of existence of an NSA.

The solid line denotes the occurrence of a Hopf bifurcation in the averaged equations. It gives a good approximation to the boundary of onset of chaotic behaviour.

The approximations are equally robust when viewed in the three-dimensional parameter space obtained by varying λ. Fig. 5 shows the regions of chaotic and non-chaotic behaviour plotted against λ for fixed values of F, Ω, together with the values of λ at which the quasi-periodic solution (3.1) ceases to exist and at which the Hopf bifurcation in the averaged equations occurs. Again the agreement is good.

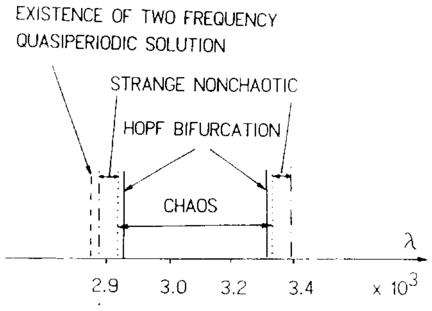

Fig. 5. Domains of strange chaotic and strange non-chaotic attractors together with the boundary of existence of the solution (3.1) and Hopf bifurcation: $F=1.1\times10^8$, $\Omega=1500$ (point c in fig. 1).

The coincidence of the NSA in parameter space with the region of lowest subharmonic response in the results of Qin et al. suggests that the other "tongues" of NSA behaviour may be associated with the cessation of existence of other, "subharmonic", quasi-periodic solutions; we have not as yet examined this conjecture.

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