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Existence and Characterization of Strange Nonchaotic Attractors in Nonlinear Systems

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Abstract—Evidence has accumulated in recent years of the occurrence, in certain nonlinear systems, of strange nonchaotic attractors, that is attractors whose geometrical character is not simple, but on and near to which the exponential divergence of trajectories, characteristic of chaotic behaviour, does not occur. This behaviour has implications for predictability; small errors in initial conditions grow much more slowly than in a chaotic system.

Such attractors occur commonly in quasiperiodically forced nonlinear oscillators, where their range of existence in parameter space is substantial; we describe two particular cases, one restricted to mechanics, the other to chemistry. Long nonchaotic transients occur in other system.

Most evidence for strange nonchaotic attractors arises from numerical experiments, and certain spectral features have been proposed [F. Romeiras and E. Ott, *Phys. Rev.* A35, 4404 (1987)] as distinguishing characteristics. Some analytical methods are also indicated which give plausible and, as compared with numerical results, quite accurate bounds in parameter space for their existence.

1. INTRODUCTION AND BACKGROUND

The existence of strange attractors in dynamical systems has been recognised at least since the celebrated paper of Lorenz [1], and was suspected well before that, though the term itself appears to have been used first only in 1971 by Ruelle and Takens as a broad description [2]. The essential characteristic of a strange attractor is best defined in negatives, as for example in Romeiras and Ott [3]. Thus a strange attractor is one which is not: a finite set of points; a limit cycle (closed curve); a smooth (piece wise smooth) surface; and bounded by a piece wise smooth closed surface [3–6].

This description relates to the geometrical character of the attractor, an object in phase space towards which trajectories are drawn as time approaches infinity. It says nothing more about the trajectories themselves, either singularly or as a class. In particular, it says nothing about the rate of divergence of two trajectories 'initially close together'. Nevertheless, for some time, the concepts of strangeness in geometrical character and of exponential divergence of trajectories, implying great sensitivity to initial conditions or chaos, and hence poor predictability, were widely regarded as almost synonymous.

Recently several instances have been described of nonchaotic strange attractors [3–10], and, in some classes of forced nonlinear systems, they appear to be of normal occurrence, that is, they persist over substantial ranges of parameter values. Such attractors are undoubtedly strange, but the behaviour of neighbouring trajectories is no longer exponentially divergent for large time even though each trajectory may have an arbitrarily complex form. Thus the outlook for *predictability* of the state evolving from a given initial condition

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is much improved; small errors in describing the initial stage arising, for example, from inaccurate observations, or from over-crude approximation, need not lead to dramatically false results in a finite time.

This difference in qualitative behaviour motivates the identification of nonchaotic strange attractors (NSAs), and it is convenient first of all to rehearse briefly the methods by which the existence of strange attractors, chaotic or nonchaotic, may be deduced. In practice, basic information usually comes in the form of time series representing the evolution of some set of observables, u, as functions of time. Conceptually we may think of a phase flow

$$u_t(x_0) = u(t, x_0)$$

constituting the solution of a continuous system, for example the autonomous ordinary differential equation

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad \text{where } \mathbf{x} \in \mathbb{R}^n$$

or alternatively a set of iterates, x_n of a discrete system, for example a one-dimensional map

$$T: x \rightarrow f(x).$$

Real data, relating even to a continuous system will of course often consist of a set of values of a smooth function, u, obtained at a sequence of specific times, T, $T + \tau$, $T + 2\tau, \ldots, T + n\tau, \ldots$

$$u_0(T) \rightarrow u_1(T + \tau) \rightarrow \ldots \rightarrow u_n(T + n\tau) \rightarrow \ldots;$$

even if this is not the case it may prove useful to generate discrete data from continuous flows by suitable Poincaré sections.

In cases where the information consists of a single time series, a basic step is to calculate a frequency spectrum; in practice this is accomplished by use of an FFT algorithm and appropriate smoothing techniques.

Knowledge of the spectrum, together with judicious use of Poincaré map information, gives much insight into the nature of an attractor. In particular, Romeiras and Ott [3] have suggested that NSAs yield spectra having a characteristic signature; if the number of spectral components larger than some value, σ is given by $N(\sigma)$, then $N(\sigma) \sim \sigma^{-\alpha}$. This contrasts with values $N(\sigma) \sim \log(1/\sigma)$, $N(\sigma) \sim [\log(1/\sigma)]^2$ for two-frequency or three-frequency quasi-periodic attractors respectively.

More direct evidence of divergence or non-divergence of neighbouring trajectories is obtained from the Lyapunov exponent. Positive Lyapunov exponents correspond to divergence; they give a measure of the rate of growth of small perturbations to a given trajectory, and therefore some measure of predictability. It is generally assumed that the existence of one or more positive Lyapunov exponents is a necessary and sufficient condition for the existence of chaos. Negative Lyapunov exponents imply decay of perturbations and hence, in general, convergence of trajectories onto attractors of lower dimension than the total phase space.

Methods of calculating the Lynapunov exponents, either from known differential equations or from experimental data, are well described in the literature (see, e.g. Wolf and Swift [11], Wolf *et al.* [12]), and their relative cheapness and straightforwardness mean that they have been the most popular indicators of chaos. It has recently been pointed out, however, by one of us (Kapitaniak and El Naschie [13], Kapitaniak [14]) that the Lyapunov exponents derived from time series data may give misleading results in the case of NSAs and that a more sensitive approach may be necessary. In the case studied the authors demonstrated that the information dimension d_1 provided a much more sensitive test. Here

$$d_I = \lim_{\varepsilon \to 0} I(\varepsilon)/\ln(1/\varepsilon) \quad \text{with} \quad I(\varepsilon) = -\sum_{i=1}^{N(\varepsilon)} p_i \ln(p_i)$$

where p_i is the measure of the attractor in the ith 'cube' of a covering of the attractor by a Cartesian grid of spacing ε, and $N(\varepsilon)$ is the total number of cubes. Effectively p_i is obtained by estimating the fraction of time the trajectory spends in the ith cube.

Much evidence has now been assembled [3–10] indicating the 'normal' occurrence of NSAs is quasi-periodically forced systems. In Section 2 we present two examples of results, for a quasi-periodically forced Van der Pol equation, and for a similarly forced system describing an autocatalytic chemical reactions.

Forcing at (at least) two irrationally related frequencies is common in engineering systems; indeed forcing at a single frequency is likely to be the exception rather than the norm. *A fortiori*, in naturally occurring dynamical systems, physical or biological, a multi-peaked spectrum of forcing is to be expected. We have examined in Section 2 the sensitivity of a nonchaotic strange attractor, arising from quasi-periodic forcing, to the presence of a perturbatory third-frequency forcing. Its robustness suggests that the qualitative behaviour obtained for two frequency forcing will hold for multi-frequency forcing, and will therefore be observed commonly in real systems. In particular we expect to see NSAs in higher order coupled systems of nonlinear oscillators with irrationally related intrinsic frequencies.

Theoretical understanding of the phenomenon of NSAs is incomplete, though Romeiras and Ott [3] have argued plausibly that a condition for their existence is the existence of a globally attracting three-torus in the essentially four-dimensional phase space of equation:

$$d^2x/dt^2 + \nu dx/dt + \sin x = K + V[\cos \omega_1 t + \cos \omega_2 t]. \tag{1.1}$$

All trajectories approach this torus, and we may expect a dynamics on the torus described by

$$d\Phi/dt = S(\Phi, t)$$

where the t dependence of S is quasiperiodic.

The strange nonchaotic NSAs reside on this torus (their strangeness has been established in the case of the quasi-periodically forced pendulum), and chaos results from a fracturing of the torus at an appropriate parameter value. A stroboscopic section of these trajectories on the three-torus at times T, $2T$, $3T, \ldots nT, \ldots$ yields a two-torus, on which the dynamics may be described by an invertible map

$$\Phi = G(\Phi, \tau), \ \tau = (\tau + 2\pi\omega/\Omega) \bmod (2\pi).$$

This map has been shown in Ref. [3] to give rise typically to NSAs.

A resemblance is apparent between this scenario and the similar occurrence of two-tori in the three-dimensional phase space of equation (1.1) for $\omega_2 = 0$ (simple forcing). Quasi-periodic behaviour on such two-tori might be expected to be broadly analogous of non-chaotic strange behaviour on the three-tori above. This expectation is born out by the numerical results of Section 2; the examples exhibit a remarkably close correspondence between the regions of parameter space occupied respectively by quasi-periodic and non-chaotic strange behaviour in the two cases.

2. QUASI-PERIODICALLY FORCED SYSTEMS

We have alluded in Section 1 to the ubiquity of NSAs in quasi-periodically forced systems. In this section we describe two specific examples, a nonlinear oscillator of Van der

Pol type, and a simple autocatalytic chemical reaction model [15]. Some earlier results on the first system have been described elsewhere [8, 10]; the second system has not, as far as we know, been investigated.

2.1. *Quasi-periodically forced Van der Pol oscillator*

We consider the equation

$$\ddot{x} - 2\lambda(1 - \beta x^2)\dot{x} + \omega_0^2 x = F\cos\omega t\cos\Omega t$$
$$= (F/2)[\cos(\omega - \Omega)t + \cos(\omega + \Omega)t]. \tag{2.1}$$

This equation, describing an oscillator with nonlinear damping, has a stable limit cycle for $\beta > 0$, implying the existence of robust finite amplitude free oscillations. It has been used in innumerable models of mechanical, electrical and chemical and biological systems. Forcing, when included, has usually been assumed to be at a single frequency. Inevitably, real systems endure forcing at two or more frequencies, and equation (2.1) is a simple example of this.

Detailed numerical evidence of the existence of an NSA in this system has been given elsewhere [8, 10], and its range of existence in (F, Ω) space for given values of λ, β, ω_0, ω is shown in Fig. 1. The corresponding largest Lynapunov exponent is shown in Fig. 2, with regions of NSAs indicated; in these regions *no* positive exponent exists, but the behaviour

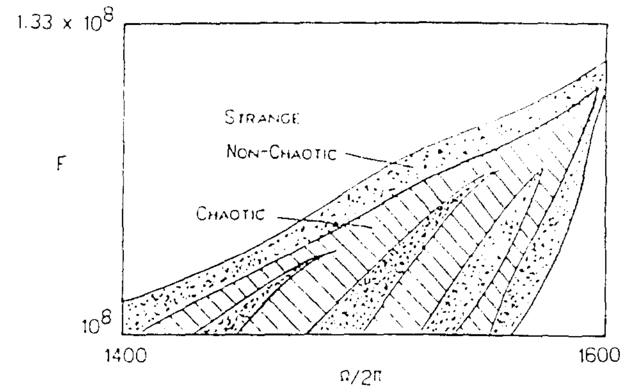


Fig. 1. Domains of strange chaotic (hatched) and strange nonchaotic (dotted) attractors of equation (2.1); $\lambda = 3 \times 10^3$, $\beta = 2.4 \times 10^{-3}$, $\omega_0 = 5550$, $\omega = \sqrt{2}/10$.

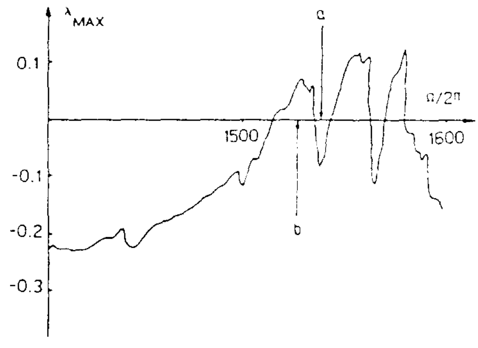


Fig. 2. The largest nonzero Lyapunov exponent λ_{max} vs Ω; $F = 1.15 \times 10^8$.

is not quasi-periodic. Not only is this deducible from the numerics; we have established analytically [9, 10] that no response of the form

$$x = A\cos(\omega t + \varepsilon_1)\cos(\Omega t + \varepsilon_2) \tag{2.2}$$

is possible.

Note that the region of existence of NSA is extensive in the parameter space (Fig. 3), so that we should expect to observe such behaviour in a real system modelled by equation (2.1).

As a test of the robustness of the behaviour to further complexities in the forcing, we have tested the response of the system when a third irrationally related frequency is added to the forcing function. The results are shown in Fig. 4, and relate to the equations

$$\ddot{x} - 2\lambda(1 - x^2)\dot{x} + x = a\cos\Omega t\cos\omega t + b\cos\bar{\omega}t \tag{2.3}$$

and

$$\ddot{x} - 2\lambda(1 - x^2)\dot{x} + x = a\cos\omega t\cos\Omega t\cos\bar{\omega}t \tag{2.4}$$

respectively.

In each case regions of existence of a nonchaotic strange attractor survive, and indeed, with three frequencies, even when the third frequency arises at the level of the small perturbation, we find that it is impossible to have anything other than chaos or NSA. In the case where $b < a$ the power spectrum is typically like Fig. 5, suggesting a 'noisy' torus, on which neighbouring trajectories do not diverge exponentially. We conjecture that such a noisy quasi-periodic behaviour will be the typical response of a real system to quasi-periodic forcing.

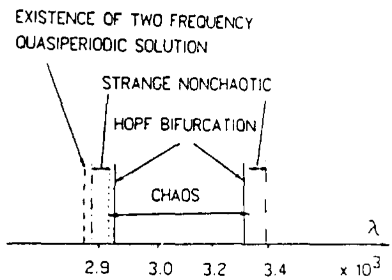


Fig. 3. Domains of strange chaotic and strange nonchaotic attractors of equation (2.1): $F = 1.1 \times 10^8$, $\Omega = 1500$.

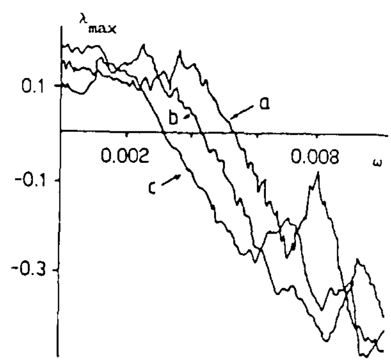


Fig. 4. The largest nonzero Lyapunov experiment λ_{max} for equations (2.3) and (2.4): $\lambda = 2.5$, $a = 5$, $\Omega = 2.64$; (a) two frequency forcing: $\bar{\omega} = 0$, $b = 0$; (b) equation (2.3): $\bar{\omega} = \sqrt{3}/10$, $b = 0.1$; (c) equation (2.4) $\bar{\omega} = \sqrt{3}/10$.

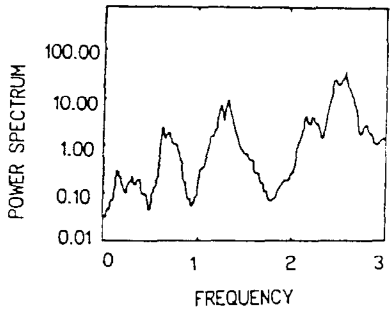


Fig. 5. Power spectrum of the response of equation (2.3): $\lambda = 2.5$, $a = 5$, $\Omega = 2.64$, $\omega = \sqrt{2}/10$, $\bar{\omega} = \sqrt{3}/10$, $b = 0.1$.

2.2. *Quasiperiodically forced chemical system*

Our second example, which, because of its relative unfamiliarity we describe more fully, is a chemical system. We consider a model for cubic autocatalysis, developed by Gray and Scott [15], for a simple reaction in a closed vessel which converts the reactant P to the product C via the intermediates A, B, according to the scheme

$$\begin{aligned} &P \rightarrow A && \text{rate} = k_0 p \\ &A + 2B \rightarrow 3B && \text{rate} = k_1 a b^2 \\ &B \rightarrow C && \text{rate} = k_2 b. \end{aligned} \tag{2.5}$$

In the type of chemical system envisaged the concentration of the initial reactant P is many orders of magnitude greater than the maximum concentrations attained by the intermediates A, B, with (to be consistent) the rate of conversion from P to A being relatively slow in comparison with the other reaction rates k_1, k_2. Consequently we can, to a good approximation, regard the concentration P as being constant and equal to its initial value p_0, i.e. we are making the 'pooled chemical' approximation.

The differential equations governing the reaction scheme then become

$$\begin{aligned} da/dt' &= k_0 p_0 - k_1 a b^2 \\ db/dt' &= k_1 a b^2 - k_2 b \end{aligned} \tag{2.6}$$

where ab are the concentrations of AB respectively and t' is time. Equations (2.8) are made non-dimensional by writing

$$a = x(k_2/k_1)^{1/2}, \; b = y(k_2/k_1)^{1/2} \quad \text{and} \quad t' = k_2 t$$

so that equations (2.6) become

$$\begin{aligned} dx/dt &= \mu - xy^2 \\ dy/dt &= xy^2 - y \end{aligned} \tag{2.7}$$

where

$$\mu = (k_0 p_0/k_2)(k_1/k_2)^{1/2}$$

is a constant of order unity.

Equations (2.6) were originally proposed by Sel'kov [16] as a simple model for oscillations in glycolysis, and their solution has been considered for all $\mu > 0$ in some detail in Ref. [18]. Strictly, equation (2.5) implies that p decays very slowly requiring μ to be proportional to $\exp(-\alpha t)$ $(\alpha > 1)$. This scheme has been studied by Merkin *et al.* [18] where a full discussion of the application of such a scheme to chemical reactions is given. Our basic assumption is to set α to be zero.

If we assume μ to be oscillating quasi-periodically about μ_0 with frequencies ω, Ω, in an attempt to model the effects on the reaction due to fluctuating external conditions, we consider the system

$$\begin{aligned} dx/dt &= \mu_0(1 + \varepsilon \cos \Omega t \cos \omega t) - xy^2 \\ dy/dt &= xy^2 - y. \end{aligned} \tag{2.8}$$

A typical set of results is shown in Fig. 6, which should be compared with Fig. 7 obtained by Merkin *et al.* [19] in a study of the simple sinusoidally forced reaction (i.e. forcing function proportional to $\cos \omega t$). Again note the close, but not exact, correspondence between regions of quasi-periodicity in the simply forced oscillator and regions of existence of an NSA in the quasi-periodically forced case.

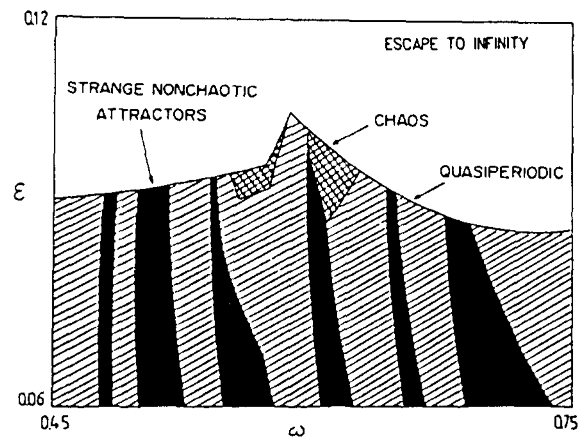


Fig. 6. Domains of chaotic (cross-hatched) and strange nonchaotic (black) attractors of equation (2.8); $\mu_0 = 0.95$, $\omega = \sqrt{2}/10$.

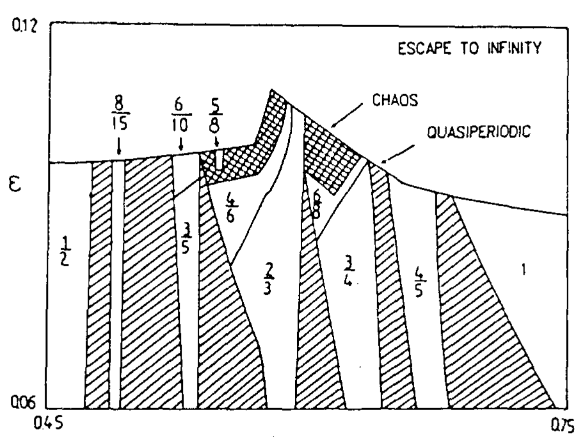


Fig. 7. Domains of chaotic (cross-hatched) and quasiperiodic behaviour of equation (2.8); $\mu_0 = 0.95$, $\omega = 0$.

3. TRANSIENT STRANGE NON-CHAOTIC BEHAVIOUR IN PERIODICALLY FORCED SYSTEMS

We have seen that NSAs are common in quasi-periodically forced systems. In this section we show that *transient* strange nonchaotic behaviour can occur also in periodically forced systems.

Consider the parametrically excited Duffing equation

$$\ddot{x} + a\dot{x} - (1 + b\cos\Omega t)x + cx^3 = 0 \tag{3.1}$$

where a, b, c and Ω are constant (Ariaratnan *et al.* [19]). Examples of this equation are found in many applications of mechanics, particularly in problems of dynamic stability of elastic systems.

Equation (3.1) has three Lyapunov exponents; one of them is always 0, one is always negative, and the third can change sign with the change of system parameters. This one can be called the largest non-zero Lyapunov exponent, λ. It is plotted in Fig. 3.1 as b changes from 0 to 0.5. When b is small, λ is negative and the system (3.1) does not show sensitive dependence on the initial conditions. When b is about 0.348, λ changes suddenly from negative to positive values and the behaviour of the system becomes chaotic.

The winding number fulfils the relation

$$w = \lim_{t \to \infty} [\alpha(t) - \alpha(0)]/t = (l/n)\Omega \tag{3.2}$$

where: $(x, \dot{x}) = (r\cos\alpha, r\sin\alpha)$; l, n are integer only, or $b < 0.256$. In the interval $0.256 < b < 0.348$ we have aperiodic motion without sensitive dependance on the initial conditions, which we call transient strange non-chaotic behaviour. Since in 3-dimensional phase space the combination of Lynapunov exponents $(0, -, -)$ guarantees ultimate approach to a limit cycle, so the observed behaviour has to be transient. This transient behaviour differs from chaotic transient behaviour in that the nearby orbits do not diverge exponentially. Time evolution towards a limit cycle seems to be following

chaotic transient	$\rightarrow$ strange non-chaotic transient	$\rightarrow$ limit cycle.
$\lambda > 0$	$\lambda < 0$	

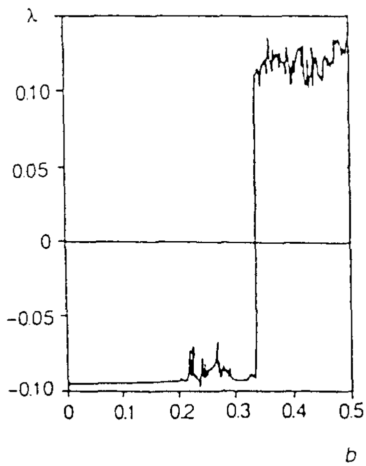


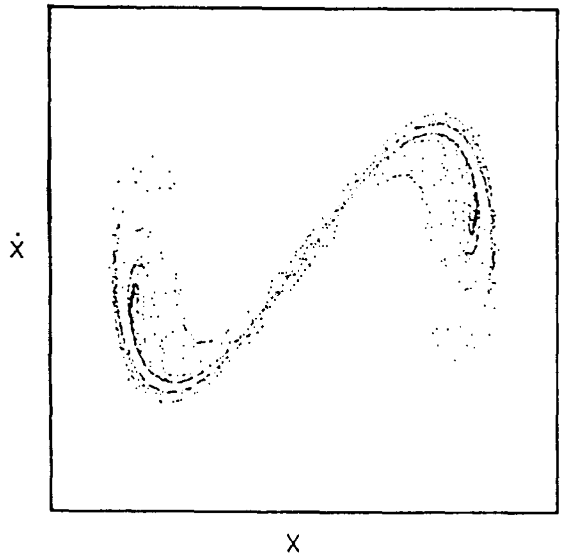
Fig. 8. The largest nonzero Lyapunov exponent λ vs amplitude of the parametrical excitation b for equation (3.1).

The Poincaré maps for the parameter value close to the boundary between transient strange nonchaotic behaviour and chaotic attractors are shown in Fig. 9(a,b).

The system (3.1) has three equilibrium positions, at $x = \pm 1$, 0 and $\dot{x} = 0$. Depending on the initial conditions it can exhibit oscillations around one of the two stable equilibria $x = \pm 1$, $\dot{x} = 0$, a small 'orbit', or around all three equilibria, a large 'orbit' (see Fig. 10).

In Fig. 11 we show the plot of maximum deflection X from equilibria $x = \pm 1$, $\dot{x} = 0$ in the case of motion on the small orbit, and from equilibrium $x = 0$, $\dot{x} = 0$ in the case of motion on the large orbit.

For the initial conditions leading to the oscillations on the small orbit it is found that this type of oscillation exists up to $b = 0.308$, when we have a sudden transition to the motion



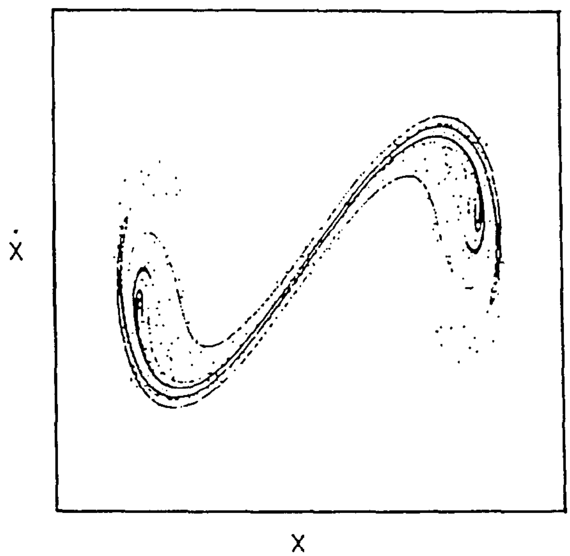


Fig. 9. Poincaré maps; (a) transient, $t = 10^5T$, $b = 0.34$; (b) $b = 0.35$.

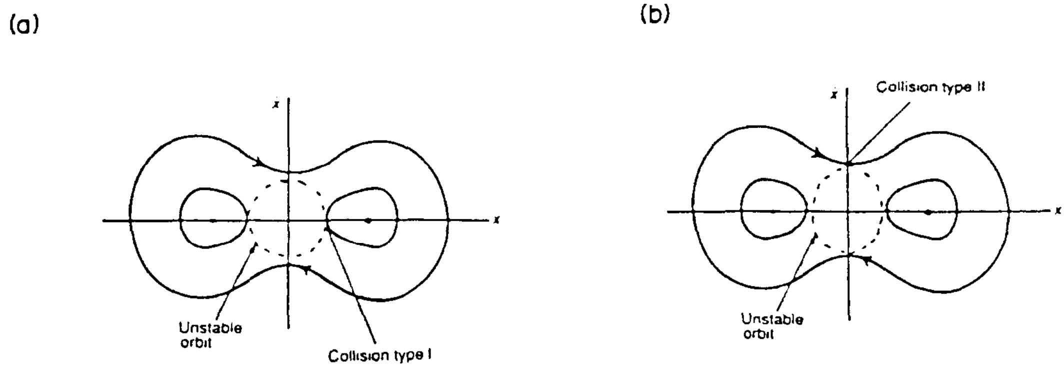


Fig. 10. The large and small orbits of the system (3.1); (a) I type collision, (b) II type collision.

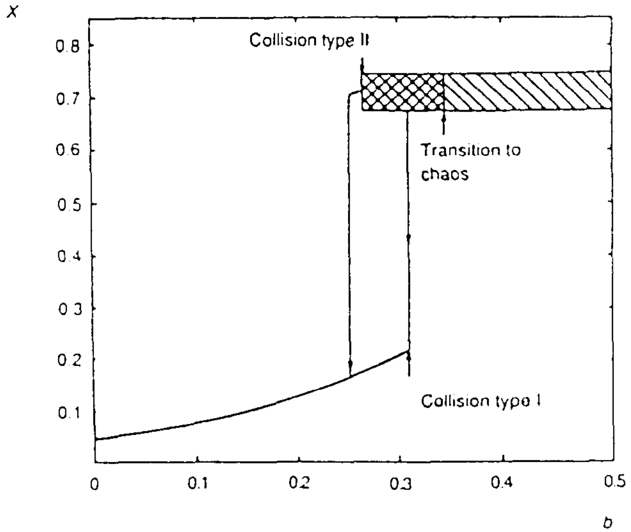


Fig. 11. Maximum deflection X vs amplitude of excitation b for equation (3.1); the 'hatching' indicates strange nonchaotic transient behaviour, and the 'cross-hatching' chaotic behaviour.

around three equilibria with a long aperiodic transient. This sudden transition is connected with the collision of the small orbit with a non-stable orbit around $x = 0$, $\dot{x} = 0$. We call this event a type I collision. Next, at $b = 0.348$, we observe a transition to chaotic behaviour.

For the initial conditions for which the large orbit is possible, we find that this orbit is stable only for $0.248 > b > 0.256$. For $b > 0.248$, only the oscillations on the small orbit are stable. At $b = 0.256$ we observe the collision of the large orbit with the unstable orbit around $x = 0$, $\dot{x} = 0$, a type II collision, and for larger b we observe transient strange non-chaotic behaviour. As in the first case, for $b \leq 0.348$, we observe transition to chaotic behaviour.

We can also fix the value of b to correspond to motion on a strange non-chaotic attractor, and change the values of a. The plot of deflection X versus a is shown in Fig. 12. As a increases from 0.1 to 0.181 we observe periodic motion on the small orbit. At $a = 0.181$ a collision of type I occurs and transient strange nonchaotic behaviour ensues.

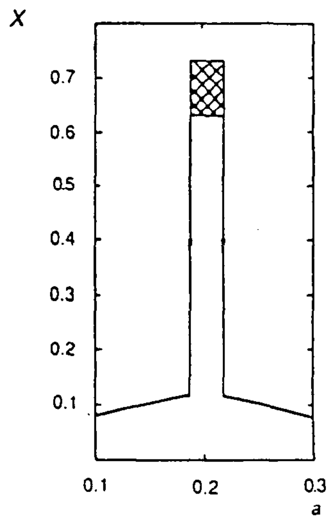


Fig. 12. Maximum deflection X vs damping coefficient a for equation (3.1).

this existing up to $a = 0.212$, when another type I collision takes place, and we have a return to periodic oscillations on the small orbit.

4. IDENTIFICATION AND PREDICTION OF NONCHAOTIC STRANGE ATTRACTORS

The results we have presented in earlier sections have all been obtained, as have those of other workers, by direct numerical computation of indicators. Markedly different spectral characteristics separate NSA behaviour from quasi-periodic behaviour, and calculation of Lyapunov exponents provides further evidence, reliable when the basis of calculation is a known mathematical system, but to be used with care when the only source of information is actual data. In this latter case, which is of course likely to be the usual one in situations of real physical concern, secure evidence is obtainable only by use of much more computationally demanding dimension calculations.

The practical indications for predictability associated with the existence of NSAs makes some form of analytical prediction highly desirable; we have found that two approaches yield promising results [8, 10].

Firstly we can find limits on parameter values for the existence of a purely quasi-periodic response to quasi-periodic forcing. The method uses a theorem first proved by Urabe [20], and an example of the resulting predicted boundary in parameter space is given in Fig. 13.

Briefly the theorem quarantees the existence of a quasi-periodic solution (2.2) of the equation (2.1) if the inequality

$$C \leq (1/52\xi)^{1/2})\{(\omega_0^2 - \lambda^2)/(2 + 2\lambda)\}^{1/4} \tag{4.1}$$

is satisfied, where the constant, C is given by

$$C = \max\{(F/2)/|\omega_0^2 - (\Omega - \omega)^2| + (F/2)/|\omega_0^2 - (\Omega + \omega)^2|,$$
$$(F/2)(\Omega - \omega)/|\omega_0^2 - (\Omega - \omega)^2| + (F/2)(\Omega + \omega)/|\omega_0^2 - (\Omega + \omega)^2|.$$

This result may be extended to establish conditions for the existence of higher harmonic quasi-periodic responses as follows. If the condition (4.1) does not hold, i.e. if there is no simple quasi-periodic solution of the form (2.2), we can expect more complicated

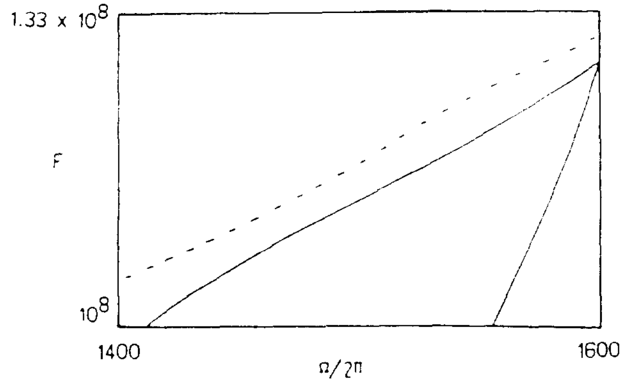


Fig. 13. Boundary of existence of the solution (2.2) (broken line) and boundaries of Hopf bifurcation (solid line) for equation (2.1). (system parameters as in Fig. 1).

quasi-periodic solutions of the form

$$x(t) = \bar{B}(0, 0) + \sum_{r=1}^{m} \sum_{|p|=r} \{\bar{C}_p \cos(p, \nu)t + \bar{D}_p \sin(p, \nu)t\} \tag{4.2}$$

where $(p, \nu) = p_1\nu_1 + p_2\nu_2$, $|p| = |p_1| + |p_2|$, $\nu_1 = \Omega - \omega$, $\nu_2 = \Omega + \omega$, the unknown coefficients $B(0, 0)$, $C_p B_p$ can be determined by any approximate method (for example Galerkin).

For the computed approximation $\bar{z}(t) = (\bar{x}(t), \dot{\bar{x}}(t))^T$

$$x(t) = \bar{B}(0, 0) + \sum_{r=1}^{m} \sum_{|p|=r} \{\bar{C}_p \cos(p, \nu)t + \bar{D}_p \sin(p, \nu)t\}$$

we have the residual function

$$R(t) = \ddot{\bar{x}} - 2\lambda(1 - \beta\bar{x}^2)\dot{\bar{x}} + \omega_0^2\bar{x} - F(\cos\nu_1 t + \cos\nu_2 t).$$

Expanding $R(t)$ in finite double Fourier series one obtains

$$R(t) = b(0, 0) + \sum_{r=1}^{3m} \sum_{|p|=r} \{c_p \cos(p, \nu)t + d_p \sin(p, \nu)t\}.$$

Next define

$$r = |b(0, 0)| + \sum_{r=1}^{3m} \sum_{|p|=r} \{|c_p| + |d_p|\}$$

have

$$\forall_{t\in J} |R(t)| \leq r.$$

If we now introduce

$$T = \bar{B}(0, 0) + \sum_{r=1}^{m} \sum_{|p|=r} (|\bar{C}_p| + |\bar{D}_p|)$$

and

$$T' = \sum_{r=1}^{m} \sum_{|p|=r} |(p, \nu)|(|c_p| + |d_p|)$$

then we have the inequalities

$$T \geqslant \sup |\bar{x}(t)|$$
$$T' \geqslant \sup |\dot{\bar{x}}(t)|.$$

If the solution z lies in a δ-neighbourhood of $\bar{z}(t) = [\bar{x}(t), \dot{\bar{x}}(t)]^T$ we have

$$\|\Psi(z, \lambda) - A(\lambda)\| \leqslant 2\lambda\{T(2T' + T) + 2(t' + 2T)\delta + 3\delta^2\} \tag{4.3}$$

Taking into account (4.3) we have the following results: if there exists a non-negative number $\kappa < 1$, and a positive number δ satisfying both inequalities

$$2\lambda\{T(2T' + T) + 2(T' + 2T)\delta + 3\delta^2)\} \leqslant (\lambda/2)((1 - \lambda^2)^{1/2}\kappa/2(2 + 2\lambda)^{1/2}$$
$$2r(2 + 2\lambda)^{1/2}/\lambda(1 - \kappa)(1 - \lambda^2)^{1/2} < \delta$$

then by the theorem of Urabe, the quasi-periodic solution (4.2) exists.

A second analytical condition, which apparently gives an approximation to the boundary between nonchaotic and chaotic behaviour, is obtained by seeking Hopf bifurcations in a simpler system of ordinary differential equations obtained from (2.1) by a suitable averaging procedure. The details have been fully described elsewhere, and typical results are included in Figs 3 and 13.

Aside from these two partly analytical results, detection of NSAs has been achieved through direct numerical analysis of data; calculation of spectra or calculation of some derived quantity, for example either Lyapunov exponent or some form of information dimension is required.

Romeiras and Ott [3] have proposed a method based on direct analysis of spectra. Introducing $N(\sigma)$, defined as the number of spectral components larger than some value σ, they conjecture that $N(\sigma) \sim \sigma^{-\alpha}$ for NSAs in contrast to $N(\sigma) \sim ln(1/\delta)$ for two frequencies quasi-periodic attractors and $N(\sigma) \sim \ln^2(1/\sigma)$ for three frequencies quasi-periodic attractors. Evidence for such distinctive spectral behaviour in the forced damped pendulum equation was adduced by them.

An alternative approach, based on Lyapunov exponents, has been proposed by Ditto *et al.* [21] and Kapitaniak [14]. Estimation of Lyapunov exponents is reliable when the equations driving the system are known but is not reliable when used on data obtained from a long time series of observations in a case where the *equations* are unknown. In this second case, in order to distinguish between chaotic and non-chaotic attractors, the properties of information dimension d_I are used. The presence of two-frequency quasi-periodic forcing guarantees that every attractor will be at least two-dimensional. According to Kaplan–Yorke conjecture [22] the information dimension of strange nonchaotic attractors is $d_I = 2$. In practice the dimension estimation is performed on the surface of cross section of the attractor, reducing all dimensions by one, i.e. a strange nonchaotic attractor occurs if the information dimension of an attractor on the cross section is $d_I = 1$ and a chaotic attractor occurs if $d_I \geqslant 2$. More details of this method are given in Ref. [14].

5. DISCUSSION AND SUMMARY

Our objectives in this paper have been threefold. Firstly to draw further attention to the occurrence of a type of behaviour in nonlinear dynamical systems which is complex, but in which neighbouring trajectories do not diverge exponentially, as they do when the behaviour is chaotic. We have attributed this behaviour to the existence to a nonchaotic strange attractor (NSA). Secondly we have stressed the robustness of such a behaviour, particularly in systems which are quasi-periodically forced, and thirdly we have described methods for the detection of NSAs and for the establishment of bounds in parameter space for their existence.

The distinction between an NSA and a strange attractor is likely to be important when detailed calculation of trajectories is required. Though trajectories may appear equally complex in the two cases, small errors in an initial condition will remain small for far longer if we have an NSA, and we should expect much better predictability. Though transient non-chaotic strange behaviour is possible in periodically forced systems, it seems that permanent NSAs occur widely for systems which are forced quasi-periodically and that they persist when the forcing has still more independent frequencies. Indeed, the parameter space for a system forced at multiple frequencies may well be divided into regions of chaotic and nonchaotic strange behaviour; generation and analysis of suitable data will be valuable.

Finally, several avenues of further investigations suggest themselves. Of much interest will be the examination of coupled systems of nonlinear oscillators having different intrinsic frequencies, and of (formally infinite-dimensional) fluid systems with strong modal structure forced by boundaries. We might expect to find NSAs at least in cases where one or two oscillators dominate, effectively driving the others much as the forced systems considered here. Clearer understanding of several of the 'analytical' results is also awaited, especially of the spectral signature of NSAs, introduced by Romeiras and Ott [3], and of the importance of the Hopf bifurcation, in the averaged system for the onset of chaos in the original system, which we have described in Section 4. Above all, as in many fields of dynamical systems theory, further well planned suites of numerical experiments are vital for the establishment of a database against which to test theoretical ideas.

Finally one should mention a result which was recently reported by Kapitaniak [14]. Based on a generalization of the triadic cantor set to higher dimension El Naschie showed that quasi-ergodicity is typical for four dimensional dynamical systems. The implications of this result for quasi-periodically forced oscillators and strange nonchaotic sets are obvious.

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