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LETTERS TO THE EDITOR

A NOTE ON ELASTIC TURBULENCE AND DIFFUSION

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1. INTRODUCTION

In contrast to the evolution of the velocity field of a fluid, which is governed by the Navier–Stokes partial differential equations, the trajectory of a fluid particle is governed by an ordinary differential equation. Consequently, in a Lagrangian description of a fluid, the results of dynamical systems theory for autonomous ordinary differential equations may be used directly. In this way the observed randomness of Lagrangian turbulence might be interpreted as deterministic chaos [1]. In the present work we give simple arguments based upon a classical elastic model to confirm the existence of chaotic diffusion-like particle paths in the presence of deterministic wave-like fluctuations. There are numerous analogies between elastomechanical and hydrodynamical problems, such as that holding between the shape of a free fluid surface under tension and the bending of an elastic wire. In the following, we use another analogy relating to the Euler elastica shown in Figure 1 and described in Appendix B [2–6] and to the lateral displacement of a fluid particle due to the motion of a circular cylindrical solid body in a two-dimensional flow described in Figure 2 and Appendix A [7, 8].

2. PSEUDO-RANDOM WALK OF A FLUID PARTICLE

Consider a circular cylinder of radius a moving in a liquid when its center is at the origin of a fixed Cartesian system (x, y) (see Figure 1). It is easily shown that the curvature of the path of a fluid particle which is displaced laterally by the cylinder is given by (see Appendix A)

$$\kappa = 2/a^2(2y - \eta), \tag{1}$$

where $\eta = y(1 - a^2/r^2)$. Noting that $\kappa = \Phi' = \mathrm{d}\Phi/\mathrm{d}s$, where s is the arc length of the path and Φ is the slope of the path, we may differentiate the previous equation once and find that

$$\Phi'' = (4/a^2)(\mathrm{d}y/\mathrm{d}s). \tag{2}$$

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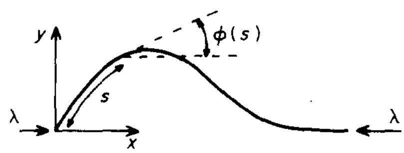

Figure 1. The elastica.

Since $dy/ds = \sin \Phi$, with $\lambda = 2/a$ one obtains

$$\Phi'' - \lambda^2 \sin \Phi = 0. \tag{3}$$

Equation (3) has obviously the same form as the classical elastica equation of Euler

$$\Phi'' + \lambda^2 \sin \Phi = 0 \tag{4}$$

(see Appendix B). One can transpose equation (3) into equation (4) by the change of variable $\Phi \to \pi + \Phi$. The elastica equation of Euler is related to the oscillations of a pendulum around its lower stable equilibrium point, while equation (3) can be related to the oscillations of a pendulum oscillating around its upper unstable equilibrium point. Both equations (3) and (4) have saddle points at $\Phi = k\pi$, $k = 0, \pm 1, \pm 2, \ldots$, and the heteroclinic orbit given by

$$\Phi = \pm 2 \arctan(\sinh s), \qquad \Phi' = \pm 2 \operatorname{sech} s. \tag{5}$$

Using the Melnikov method one can find small perturbations which when added to equations (3) or (4) can lead to the transverse intersection of stable and unstable manifolds [9]. These intersections show the possibility of chaotic behaviour (temporal in the pendulum and spatial in problems of elastica or the lateral displacement of a fluid particle due to the motion of a cylinder).

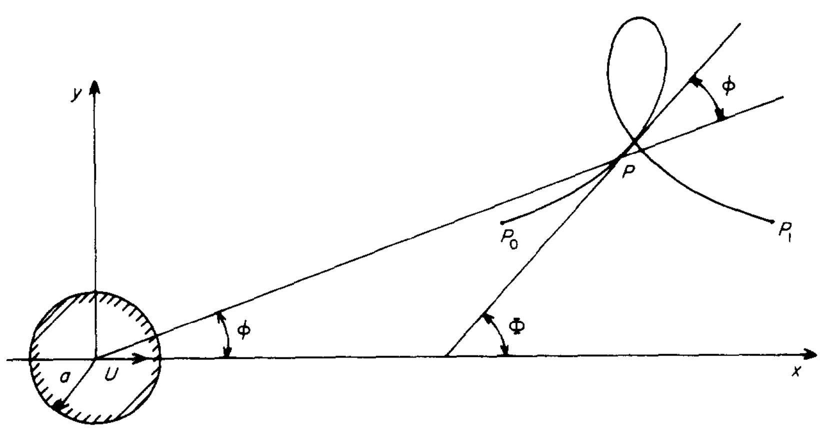

Figure 2. The path of a fluid particle laterally displaced by a rotating cylinder.

Consequently, we may state that these looped stereophoid-like paths, which were drawn long ago by J. Maxwell and G. I. Taylor [10, 11], are spatial heteroclinic orbits. In other words, they are related to spatial separatrices corresponding in a dynamical analogy to a heteroclinic orbit in phase space [4, 6, 12]. It then follows that the entire chain of reasoning used previously in establishing the possibility for statical loop soliton chaos in the Euler elastica [4–6, 12] may now be carried over to show that deterministic wave-like fluctuations could lead in the appropriate region of parameters and initial conditions to a completely chaotic particle path. The loops themselves may still persist but their spatial distribution will be erratic and will differ essentially from the classical picture given in textbooks. Now, due to the intimate relation between the stability of orbits with irrational winding numbers, damping and random walk, we feel that the preceding discussion might be relevant for the interpretation of a diffusion-like process [13].

It might even be that the analogy with the elastica can be taken one step further by using the analogy between the elastica and the elastic circular ring under external pressure [12, 14, 15]. Thus, we might expect that circular motion of a circular cylinder in a fluid will produce similar diffusion-like behaviour of the fluid particles in a circular container. However, what may be even more interesting is what could happen if we have vortices in the fluid. We discuss this next.

3. CHAOTIC VORTICES, CORNU SPIRALS AND THE FLUTTERING ELASTICA

In the previous section we considered the effect of a cylinder moving in a fluid. Here we consider two circular cylinders in a fluid stream [8]. In this case, as is well known, two kinds of streamlines form, as shown in Figure 3(a). Now as we let the radius of the cylinder

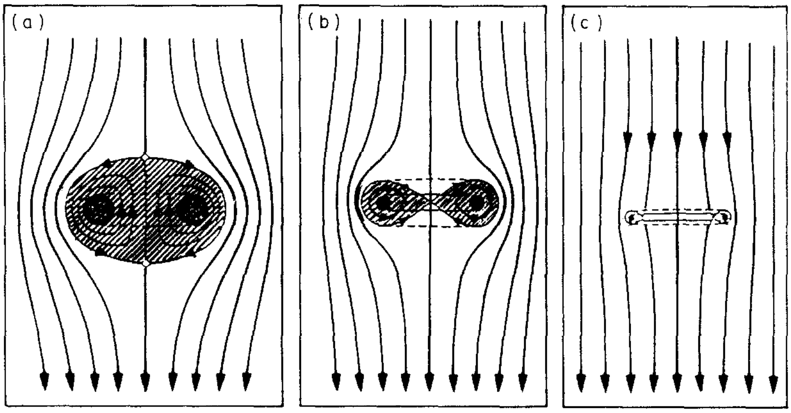

Figure 3. The evolution of streamlines of a flow with two cylinders as the radius of the cylinders shrinks [8].

shrink we obtain the sequence shown in Figures 3(b) and (c). Anyone who has observed the motion of travelling loop solitons in a long flexible wire [6, 16] will notice the similarity between them and the vortices shown in Figure 3(c), which can easily be made visible in an actual experiment. It is this similarity which was the motive for trying to model some of these fluid motions by using the elastica. In the case of a Hamiltonian system, the model was relatively straightforward and adding parametrical imperfection (see Appendix B) we found some interesting spatially chaotic deformations for equation (4) with parameters $\lambda^2=+0{\cdot}0272222$, $a=0{\cdot}15$ and $\omega=1$, as shown in Figure 4. However, some problems arise in the spatial interpretation of positively and negatively dissipative elastica. In the case of positive dissipation, i.e., damping, this may be interpreted as non-conservative tangential friction forces akin to the so-called follower forces discussed in references [5, 12]. Negative dissipation is consequently the adjoint system, the so-called flutter set [12]. We may mention that the inclusion of this type of negative damping was motivated by some problems connected to protein deformation. The results of our numerical experiments are shown in Figures 5 and 6. The spatial entanglement (Figure 5) obtained for equation (B7) ($\lambda^2=1$, $a=0{\cdot}94$, $k=0{\cdot}15$, $\omega=1{\cdot}56$), which looks quite similar to randomly coiled polymer chains, is quite interesting. They correspond in the dynamical analogy to the region of a strange attractor in a parametrically excited system [9]. However, the most interesting numerical results are those with spiral-like chaos deformation of Figure 6, obtained for equation

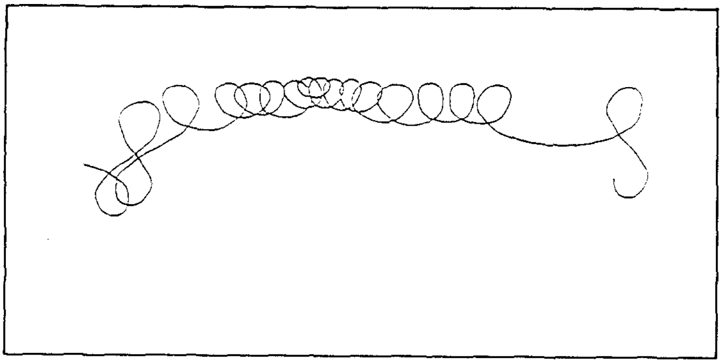

Figure 4. Spatial soliton loop chaos of the imperfect Hamiltonian elastica (B6): $\lambda^2 = +0{\cdot}0272222$, $a = 0{\cdot}15$, $\omega = 1$, $\Phi(0) = 6$, $\Phi'(0) = 0$.

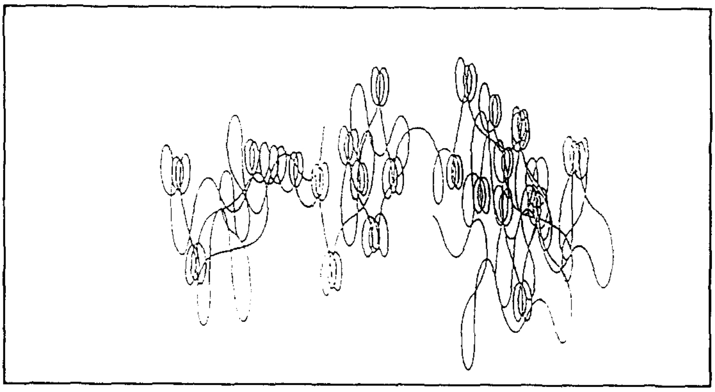

Figure 5. Spatial entanglement of the dissipative elastica (B7): $k = 0{\cdot}15$, $a = 0{\cdot}94$, $\lambda^2 = 1$, $\omega = 1{\cdot}56$, corresponding to a strange attractor in the dynamical analogy.

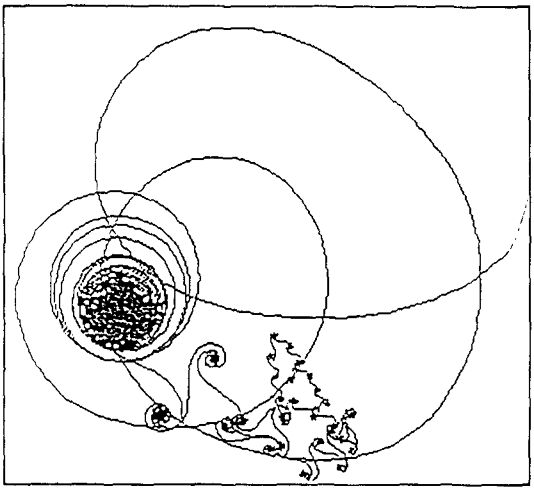

Figure 6. Loop and spiral chaos in the imperfect flutter elastica (B7): $\lambda^2 = +0{\cdot}272222$, $a = 0{\cdot}15$, $k = -0{\cdot}01$, $\omega = 1$, $\Phi(0) = 6$, $\Phi'(0) = 0$.

(B7) ($k=-0{\cdot}01$, $a=0{\cdot}15$, $\lambda^2=+0{\cdot}0272222$, $\omega=1$, $\Phi(0)=6$, $\Phi'(0)=0$). They strongly resemble some of the pseudo-random Cornu-like spirals (see Figure 5) found, for example, in the Riemann sums approximating oscillatory integrals [17]. They bear also some similarity to the Bernhard–Kármán vortices (see Figure 8) [7, 8, 10, 11]. The existence of loops on several scales shown in the computer blow-up of Figure 9 may be of special interest.

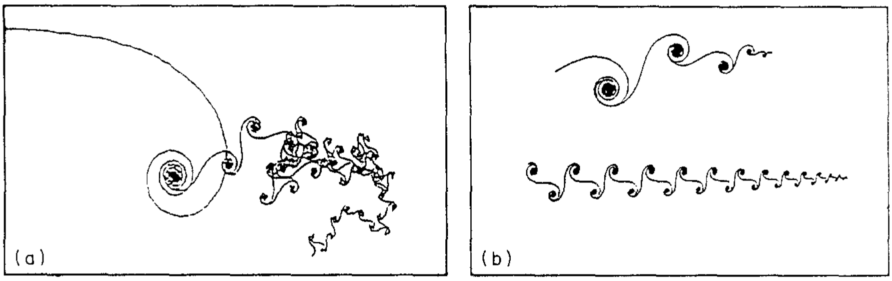

Figure 7. Regular (a) and irregular (b) Cornu-like spirals [17].

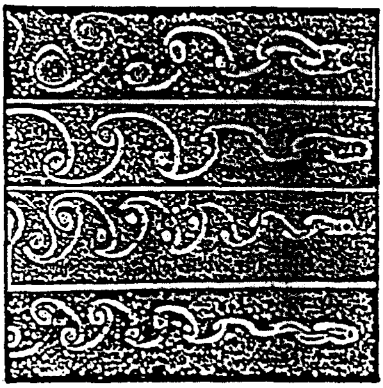

Figure 8. Some experimental studies of vortex wake.

In Figures 4–7 and 9 the spatial plots of the response of the appropriate equation (B6) or (B7) are presented. These equations have been solved numerically by the fourth order Runge–Kutta method with integration step $2\pi/200\omega$. The spatial plot has been obtained by plotting x as given by equation (B2) *vs.* y described by equation (B3).

4. CONCLUSIONS

The elastica and, in particular, the imperfect flutter elastica provides a surprisingly simple model which reflects some fundamental aspects of diffusion and turbulence-like behaviour in fluids. Of course, we can never show true chaos by using our numerical technique. Nevertheless, by using the pre-entropy related ideas of Kahlert and Rossler [19] we can show asymptotic chaos [2]. When we observed the spiral chaos of the elastica we were initially inclined to regard it as only a numerical instability phenomenon. However, repeated independent calculations by different methods have convinced us that spiral chaos is a true feature of the non-linear dynamics of our model. The appearance of self-similarity on many scales which these spirals reflect shows that we are dealing with phenomena which may be linked to mixing and diffusion-like processes.

The modification of our elastica model allows us to mimic to some reasonable extent the well known Kármán vortex street created in a fluid by the movement of the cylinder

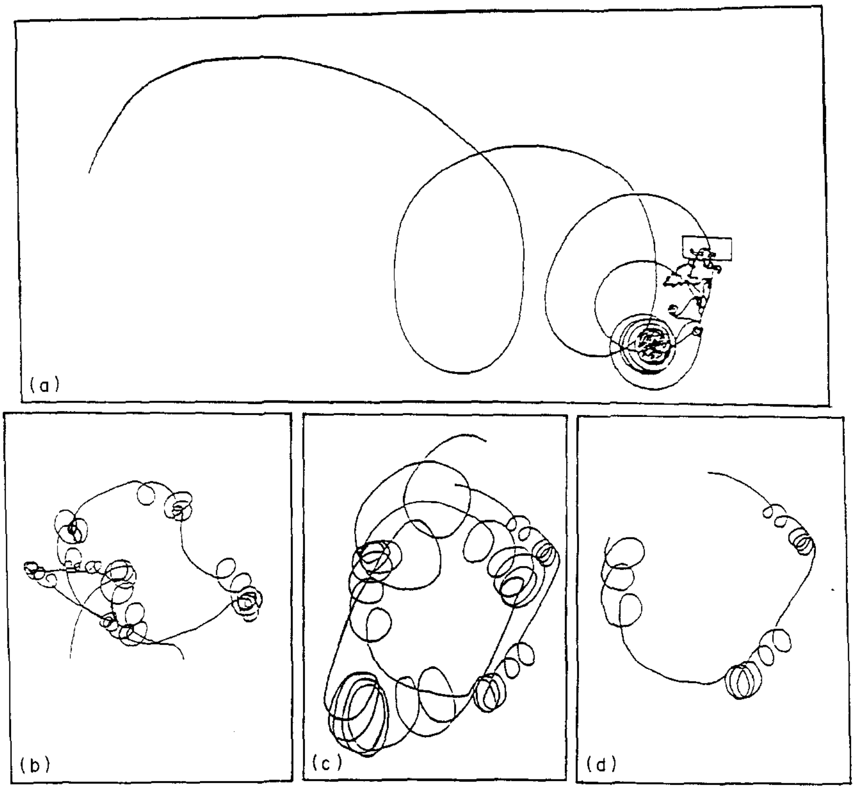

Figure 9. Computer blow-up of the turbulence spirals of the flutter elastica (B7) (λ^2, a, k, ω as in Figure 6) showing the existence of loops on many scales: (b) enlargement of the box from (a); (c, d) initial details. Note also the similarity to turtle geometry [18] and that the loop is the building block of the geometrical form.

at certain Reynold numbers [8] (see Figures 4–6). It is interesting to note that extending the elastica analogy to a circular elastica coiled infinitely many times inside a ridged confinement shows a striking resemblance to the ergodic properties of diffusion processes and billiard dynamics in a magnetic field.

Finally, we have pointed out the similarity of these vortices to travelling loop solitons [16, 20], turtle geometry [18], and the Cornu-like spiral chaos which arises in Riemann sums approximating oscillatory integrals [17] (see Figure 5).

REFERENCES

1. J. P. Eckmann 1981 *Review of Modern Physics* **53**, 643–654. Roads to turbulence in dissipative dynamical system.
2. M. S. El Naschie and S. Al Athel 1989 *Zeitschrift für Naturforschung* **44a**, 645–650. On the connection between statical and dynamical chaos.
3. A. Mielke and P. Holmes 1988 *Archives of Rational Mechanics and Analysis* **101**(4), 491–496. Spatially complex equilibria of buckled rod.
4. M. S. El Naschie 1989 *Journal of the Physical Society of Japan* **58** (12), 4310–4321. On certain homoclinic soliton in elastic stability.
5. M. S. El Naschie and T. Kapitaniak 1990 *Physics Letters* **A147**, 275. Soliton chaos models for mechanical and biological elastic chain.
6. M. S. El Naschie, S. Al Athel and A. C. Walker 1990 in *Nonlinear Dynamics in Engineering Systems* (W. Schiehlen, editor), 67–74. Berlin: Springer-Verlag. Localized buckling as statical homoclinic soliton and spacial complexity.

7. H. BERNHARD 1990 *Revue Generale des Sciences Pures et Appliquées* **11**, 1261–1271; 1309–1328. Les tourbillons cellulaires dans une nappe liquide.
8. L. PRANDTL and O. TIETJENS 1957 *Fundamentals of Hydro- and Aerodynamics*, New York: Dover. See in particular, pages 64 and 211–213.
9. T. KAPITANIAK 1991 *Chaotic Oscillations in Mechanical Systems*. Manchester University Press.
10. H. LAMB 1945 *Hydrodynamics*. New York: Dover.
11. L. MILNE-THOMSON 1960 *Theoretical Hydrodynamics*, London: Macmillan. See, in particular, pages 233–237.
12. M. S. EL NASCHIE 1990 *Stress, Stability and Chaos*. London: McGraw-Hill.
13. J. CRANK 1975 *The Mathematics of Diffusion*. Oxford: Clarendon Press.
14. M. S. EL NASCHIE 1975 *International Journal of Mechanical Science* **17**, 387. The initial post-buckling of an extensional ring under external pressure.
15. M. S. EL NASCHIE 1989 *12th ASME Conference on Mechanical Vibration and Noise* **H0508E** (T. S. Sankar, editor), 377–383. New York: American Society of Mechanical Engineers. Soliton and chaos in eigenvalue problems.
16. Y. H. ICHIKAWA, K. KONNO and M. WADATI 1983 in *Long-Time Prediction in Dynamics* (C. W. Horton *et al.*, editors) New York: Wiley-Interscience. New nonlinear evolution equation leading to exotic soliton.
17. E. COUTSIAS and N. KAZARINOFF 1987 *Physica* **26D**, 295–310. Disorder, renormalizability, theta functions and Cornu spirals.
18. H. ABELSON and A. DISESSA 1986 *Turtle Geometry*. London: The MIT Press.
19. C. KAHLERT and O. E. ROSSLER 1984 *Zeitschrift für Naturforschung* **39a**, 1200–1204. Chaos in boundary value problems.
20. F. HOMANN 1939 *Forschung Ingeinifür Wesen* **7**, 423–442. Einfluss grosser Zaehigkeit bei Stroemung um Zylinder.

APPENDIX A: THE DIFFERENTIAL EQUATION OF A FLUID PARTICLE

Consider fixed axes x, y at the instant when the center of the cylinder is at 0, and polar co-ordinates r, ϕ, the origin of which also coincides with the center of the cylinder (see Figure 1). The particle at the point $P(x, y)$ is moving with velocity Ua^2/r^2 at an angle Θ with the radius vector and therefore the tangent to the path of P makes an angle ϕ with x, where $\Phi = 2\phi$. Hence, the curvature of the path of P is given by

$$\kappa = 1/R = \mathrm{d}\Phi/\mathrm{d}s = (\mathrm{d}\Phi/\mathrm{d}y)(\mathrm{d}y/\mathrm{d}s) = (\mathrm{d}\Phi/\mathrm{d}y)\sin\Phi, \tag{A1}$$

where R is the radius of curvature and $\Phi = 2\phi$ is the slope of the path measured from the horizontal x direction of the motion. On the other hand, the streamline is given by

$$\eta = y(1 - a^2/r^2), \tag{A2}$$

and noting that $\sin\phi = \sin(\Phi/2) = y/r$ one can eliminate r and find

$$\eta = y[1 - a^2 \sin(\Phi/2)^2/y^2]. \tag{A3}$$

This may be written as

$$(\sin(\Phi/2))^2 = (1/a^2)(y^2 - \eta y), \tag{A4}$$

and differentiating both sides as a function of y one finds

$$\mathrm{d}(\Phi/2)/\mathrm{d}y = (1/a^2)(2y - \eta). \tag{A5}$$

Hence

$$(1/a^2)(2y-\eta)=\sin\Phi\ \mathrm{d}(\Phi/2)/\mathrm{d}y=0{\cdot}5\sin\Phi\ \mathrm{d}\Phi/\mathrm{d}y, \tag{A6}$$

or

$$(\mathrm{d}\Phi/\mathrm{d}y)\sin\Phi=(2/a^2)(2y-\eta), \tag{A7}$$

where the constant η gives the initial and final distances of the particle. Differentiating equation (A6) one finds

$$\Phi''=(4/a^2)(\mathrm{d}y/\mathrm{d}s). \tag{A8}$$

Noting that $\mathrm{d}y/\mathrm{d}s=\sin\Phi$ one obtains

$$\Phi''-\lambda^2\sin\Phi=0, \tag{A9}$$

where $\lambda=2/a$. Equation (A9) has the same form as the equation of the elastica (see Appendix B).

APPENDIX B: THE EQUATION OF THE ELASTICA

1. The Hamiltonian equation of the perfect initially straight elastica corresponding to an undamped pendulum is

$$\Phi''+\lambda^2\sin\Phi=0, \tag{B1}$$

where Φ is the angle of inclination of the central line of the elastica (see Figure 1), $(')=\mathrm{d}(\ )/\mathrm{d}s$ and s is the arc length of the deformed elastica which is assumed to be totally inextensible. Consequently, Φ' is the curvature of the deformed (buckled) elastica. The parameter $\lambda=\sqrt{P/\alpha}$ corresponds to the natural frequency, where P is the axial load and α is the bending stiffness. This means that the bending moment is $M=\alpha\Phi'$.

The displacement (x, y) may be calculated from the formulas

$$x(s)=\int_0^s\cos\Phi(s')\,\mathrm{d}s',\qquad y(s)=\int_0^s\sin\Phi(s')\,\mathrm{d}s'. \tag{B2–B4}$$

2. The imperfect elastica corresponding to a periodically excited pendulum is

$$\Phi''+\lambda^2\sin\Phi=a\sin\omega s, \tag{B5}$$

where a is a measure for the amplitude of the harmonic axial imperfection (crookedness) of the central line and ω is the frequency of this periodic spatial imperfection.

3. The elastica corresponding to a parametrically excited pendulum is

$$\Phi''+\lambda^2\sin\Phi=a\sin\omega s\sin\Phi. \tag{B6}$$

4. The elastica corresponding to a parametrically excited pendulum with positive (or negative) damping is

$$\Phi''+\lambda^2\sin\Phi\pm k\Phi'=a\sin\omega s\sin\Phi, \tag{B7}$$

where k is the constant of linear positive (or negative) dissipation. For more details and derivation see references [4, 12].