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STRANGE NON-CHAOTIC TRANSIENTS

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Transient behaviour has always played an important role in various areas of science. In the past two decades it has been found that strange attractors are common in the realm of non-linear systems.

Although chaos is an asymptotic property which manifests itself only after a very long time (for example, Lyapunov exponents are defined only in the infinitely long time limit) it is possible to define chaotic transients.

Let $\mathfrak{T}$ be the internal characteristic time of the system: i.e., for continuous time problems $\mathfrak{T}$ can be the average turnover time of trajectories in phase space, or, in non-autonomous systems, the reciprocal value of the driving frequency.

A necessary condition for aperiodic (random looking) transient signals of average lifetime τ_s to be strange is that they last much longer than the internal characteristic time: i.e.,

$$\tau \gg \mathfrak{T}. \tag{1}$$

For $t < \tau$ the Poincaré map of the system has a fractal structure.

If criterion (1) holds and the system is characterized by sensitive dependence on initial conditions up to a lifetime τ_c (a positive maximum Lyapunov exponent for $t < \tau_c$), this type of transient is called chaotic.

If, on the other hand, condition (1) holds and the system does not show sensitive dependence on initial conditions (a non-positive maximum Lyapunov exponent for $\tau_c < t < \tau_s$) we have a strange non-chaotic transient. Of course if $\tau_c = \tau_s$ we do not observe strange non-chaotic transients. Typical time series without strange non-chaotic transients and with them are shown in Figures 1(a) and (b).

Both transiently strange signals have the following properties. They look strange up to the time τ_s and then switch over into non-strange behaviour, which governs all the rest of the signals.

Transient chaos was first observed in the Lorenz model with fixed points [1, 2] and the limit cycle [3] as an attractor. Subsequently, many papers reported this phenomenon in various types of non-linear systems: non-linear oscillators [4, 5], delay equations [6], partial differential equations [7, 8] and coupled map lattices [9, 10].

Until now there have been no reports on strange non-chaotic transients. The possibility of the existence of this new type of transient behaviour is shown in what follows.

Consider the parametrically excited Duffing's equation

$$\ddot{x} + a\dot{x} - (1 + b\cos\Omega t)x + cx^3 = 0, \tag{2}$$

where a, b, c and Ω are constants [11]. Examples of this equation are found in many applications of mechanics, particularly in problems of dynamic stability of elastic systems [12].

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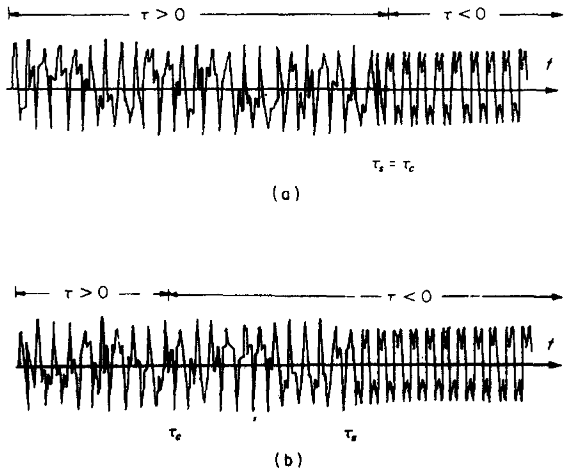

Figure 1. Time series without strange non-chaotic transient (a), and with it (b).

In the numerical calculations we have used the fourth order Runge–Kutta method with the integration step being $T/200$, where $T=2\pi/\Omega$. Lyapunov exponents have been obtained using the method of Wolf *et al.* [13].

Equation (2) has three Lyapunov exponents: one of them is always zero, one is always negative, and the third one can change its sign with changes in the system parameters. One can call it the largest non-zero Lyapunov exponent: λ. If λ is negative we have the limit cycle attractor if λ is positive we have the strange chaotic attractor and if $\lambda=0$ a torus is an attractor [14, 21]. The largest non-zero Lyapunov exponent λ for $t\to\infty$ is plotted in Figure 2 as b changes from 0 to 0·5. if b is small, λ is negative, and so the system (2) does not show sensitive dependence on the initial conditions. When b is increasing up to about 0·348, λ changes suddenly from negative to positive values, and the behaviour of the system is chaotic.

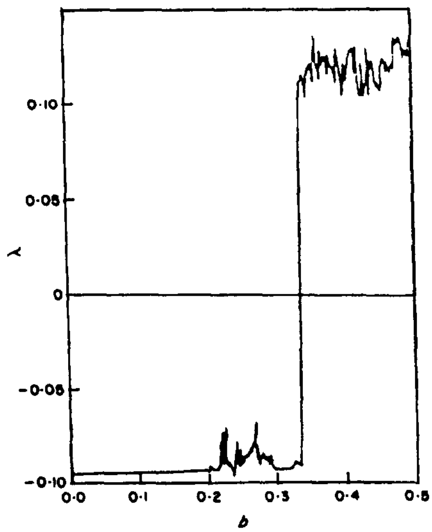

Figure 2. The largest non-zero Lyapunov exponent λ *vs.* amplitude of the parametrical excitation b.

If one considers the dependence of the winding number

$$w(b) = \lim_{t \to \infty} (\alpha(t) - \alpha(0))/t, \tag{3}$$

where $(x, \dot{x}) = (r\cos\alpha, r\sin\alpha)$, $t \gg \tau_c$, on the parameter b it is found that the relation

$$w/\Omega = l/n, \tag{4}$$

where l and n are integers, is satisfied only up to values of $b = 0{\cdot}256$. In the interval $b \in (0{\cdot}256, 0{\cdot}348)$ we have aperiodic motion without sensitive dependence on the initial conditions. As a combination of Lyapunov exponents $(0, -, -)$ indicates the limit cycle as an attractor this behaviour has to be transient. The Poincaré maps for the parameters close to the boundary between transient strange non-chaotic behaviour and chaotic attractors are shown in Figures 3(a) and (b). The map shown in Figure 3(a) is transient for $t < \tau_s$.

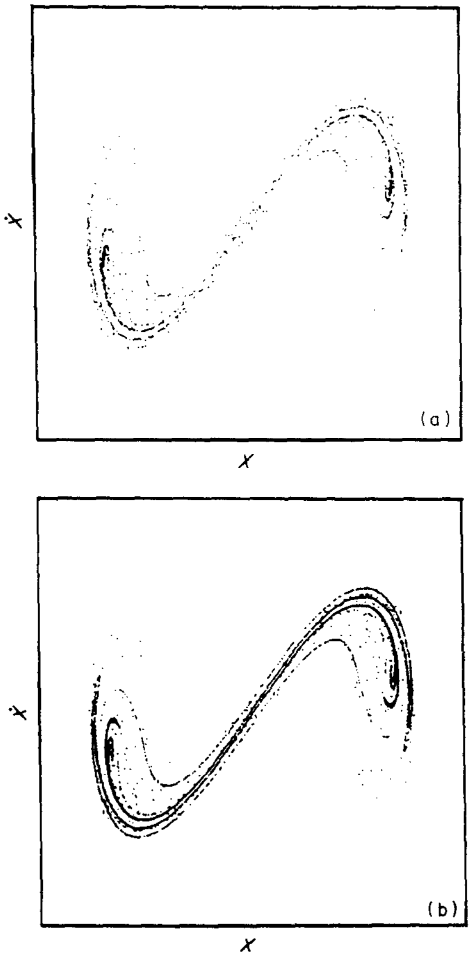

Figure 3. Poincaré maps: (a) transient, $t = 10^5 T$, $b = 0{\cdot}34$; (b) $b = 0{\cdot}35$.

System (2) has three equilibrium positions: $x=\pm 1$, 0 and $\dot{x}=0$. Depending on the initial conditions it can exhibit oscillations around one of the two stable equilibria $x=\pm 1$, $x=0$—small orbit—or around all equilibria—large orbit (see Figure 4).

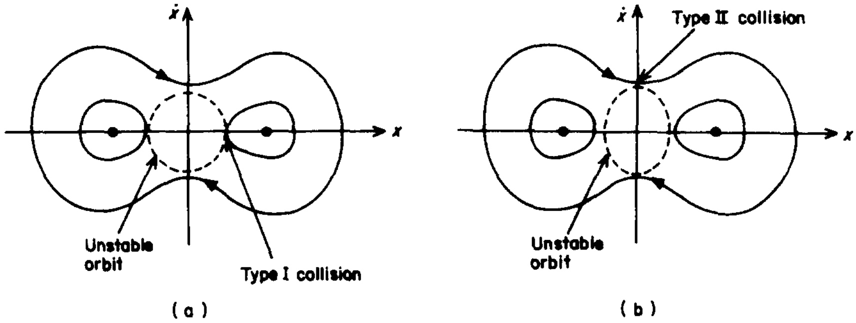

Figure 4. The large and small orbits of the system (2): (a) type I collision; (b) type II collision.

In Figure 5 we show the plot of the maximum deflection X from the equilibria $x=\pm 1$, $\dot{x}=0$ in the case of motion on the small orbit and from the equilibrium $x=0$, $\dot{x}=0$ in the case of motion on the large orbit.

For the initial conditions leading to oscillations on the small orbit it has been found that this type of oscillation exists up to $b=0{\cdot}308$, where we have a sudden transition and we observe transient motion around three equilibria, but on the aperiodic trajectory.

This sudden transition is connected with the collision of the small orbit with the non-stable orbit around $x=0$, $\dot{x}=0$. We call this event the type I collision. Next, at $b=0{\cdot}348$, the transient strange non-chaotic behaviour disappears and we have chaotic behaviour.

For the initial conditions for which the large orbit is possible we found that this orbit is stable only for $b \in (0{\cdot}248, 0{\cdot}256)$. For $b<0{\cdot}248$ only the oscillations on the small orbit are stable. At $b=0{\cdot}256$ we observe the collision of the large orbit with the unstable orbit around $x=0$, $\dot{x}=0$ (the type II collision) and for larger b the motion is characterized by

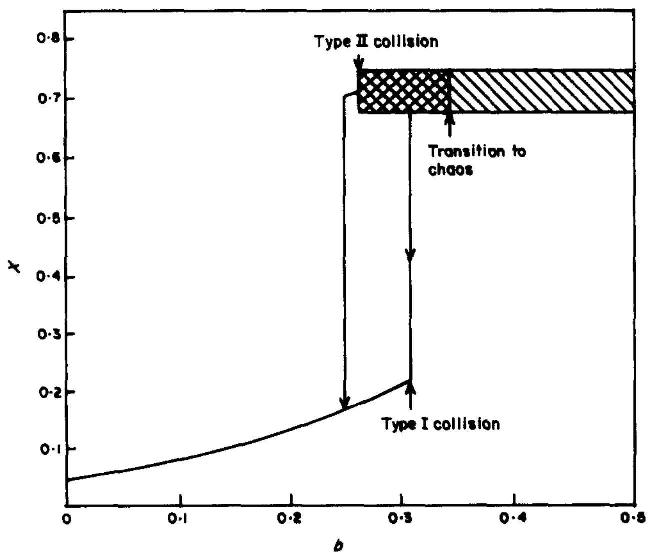

Figure 5. Maximum deflection X *vs.* amplitude of excitation b. ////, Strange non-chaotic transient behaviour; ▩ , chaotic behaviour.

transient non-chaotic behaviour. As in the first case, for $b=0{\cdot}348$ we observe transition to chaotic behaviour.

Next, we fixed the value of b equivalent to the motion with transient strange non-chaotic behaviour and changed the values of a. The plot of deflection X *vs.* a is shown in Figure 6. Increasing a from 0·1 to 0·181, we observe periodic motion on the small orbit. At $a=0{\cdot}181$ a collision of type I occurs and we have a sudden transition to the motion with transient strange non-chaotic behaviour, which exists up to $a=0{\cdot}212$ when another type I collision takes place, and we have the transition to periodic oscillations on the small orbit.

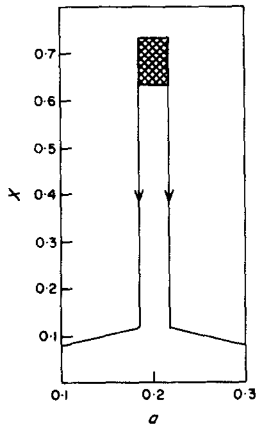

Figure 6. Maximum deflection X *vs.* damping coefficient a; $b=0{\cdot}34$.

The lifetime of the observed transient strange non-chaotic behaviour is relatively long ($\tau_s=10^6$–$10^{10}T$ have been observed). It is much longer than the chaotic transient lifetime τ^c, which has been observed to be in the range 10^2–10^3T. Both τ_s and τ_c strongly depend on the initial conditions.

Strange non-chaotic attractors have been shown to form part of the normal pattern of behaviour in quasiperiodically forced non-linear oscillators in four-dimensional phase space [17–25], and their presence has been demonstrated by a number of numerical investigations.

The evolution of our system (2) takes place in three-dimensional phase space, where strange non-chaotic attractors or even repellers are not allowed. Strange non-chaotic transients which have been found for equation (2) show the same property as typical trajectories on strange non-chaotic attractors for $t<\tau_s$.

Non-existence of strange non-chaotic repellers for equation (2) shows that strange non-chaotic transients cannot be connected with them as most chaotic transients can be connected with chaotic repellers [16].

By application of the Melnikov method it is easy to show the existence of transversal homoclinic points and horseshoe maps for

$$b>(4a/3\pi\Omega^2)\sinh(\pi\Omega/2): \tag{5}$$

i.e., for the values for which we have observed strange non-chaotic transients. The mechanism responsible for strange non-chaotic transients of equation (2) is the same as for chaotic transients (unstable strange set produced by the horseshoe map). In our case the maximum Lyapunov exponent turns negative far before the transients have died.

The appearance of strange non-chaotic transients after the collision of stable and unstable orbits of system (2) seems to be connected with the well-known property that chaotic transients typically appear in systems passing through a crisis configuration [15, 16].

It should be noted here that there is a possibility of other types of strange non-chaotic transients in at least four-dimensional phase-space, where they are produced by strange non-chaotic repellers [26].

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