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Chaos and noisy periodicity in forced ocean–atmosphere models

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We examine two models, each consisting of coupled nonlinear oscillators, which represent coupled ocean–atmosphere dynamics on climatic and seasonal timescales respectively. In each case, the addition to a single frequency forcing of an extra forcing, itself either single frequency or stochastic, is able to modify a chaotic response to one of noisy periodicity, reminiscent of actual fluctuations on ice-age or El Niño timescales.

1. Introduction

The occurrence of multiple equilibria and of chaotic behaviour in nonlinear oscillators subjected to periodic forcing is widespread and well known. An example is the Duffing equation, which we may write as

$$\ddot{x}+k\dot{x}+x^3=B\cos\Omega t\,, \tag{1.1}$$

arising almost ubiquitously in models of mechanical oscillations, which has been extensively studied [1,2]; for a useful summary see ref. [3]. The parameter space of (1.1) is divided with great complexity into regions of different qualitative behaviour, and the space of initial states is divided with similar complexity into the basins of attraction of competing attractors, which may be steady, periodic or chaotic. Solutions in chaotic regimes are often characterised by behaviour which may show some regularity over short timescales with occasional substantial deviations, as the trajectory spends time slow-moving near one unstable equilibrium between sporadic excursions to the neighbourhood of another.

Aperiodicity is the first apparent characteristic of the behaviour of many geophysical fluid dynamic systems, but atmospheric and oceanic flows often exhibit substantial coherent features, localised in either or both of space and time, which occur sporadically and unpredictably but with a certain statistical regularity. Such features are exemplified by blocking patterns in the mid-latitude atmosphere or by persistent anomalies (of which El Niño is the most spectacular) in sea surface temperature. Fig. 1 illustrates the essential character of El Niño events and also their dynamic and often seriously damaging effects on local climatic phenomena like rainfall. On a different timescale, fluctuations in climate display similar characteristics, distinguished by quasi-regular "anomalies" on timescales of decades to hundreds of thousands of years. Fig. 2 summarises in spectral form the incidence of ice ages over the past million or so years.

The existence and persistence of such features is reminiscent of the behaviour described by a solution trajectory of a nonlinear oscillator wandering near to one unstable equilibrium and making occasional excursions to the neighbourhood of a second unstable equilibrium. This similarity in behaviour has stimulated the development of simple conceptual models, usually taking the form of forced or coupled nonlin-

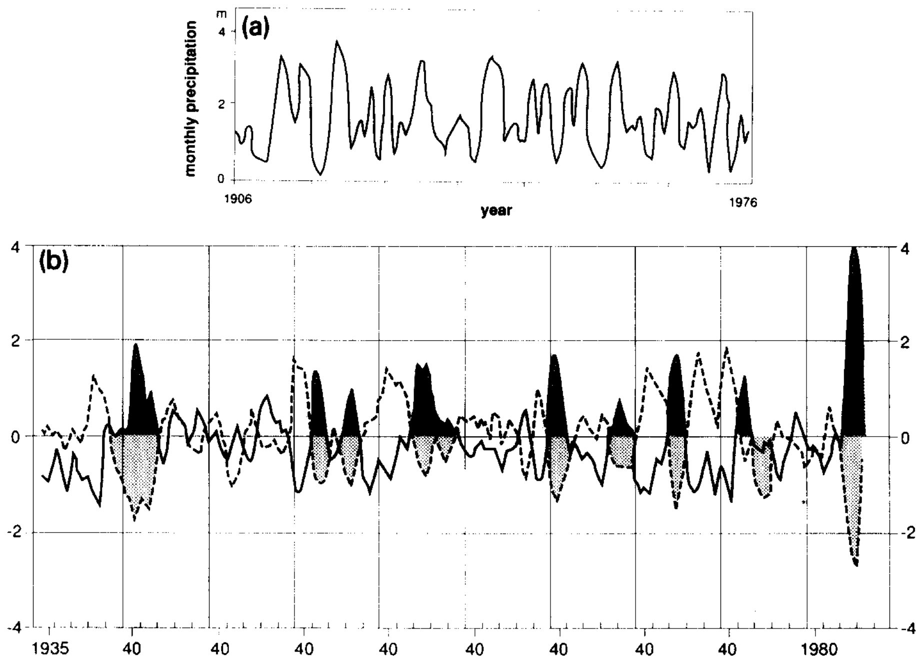

Fig. 1. (a) Mean monthly precipitation at Nauru Island. (b) Departure from mean of Eastern Pacific sea surface temperature (solid line) and southern oscillation index (dashed line).

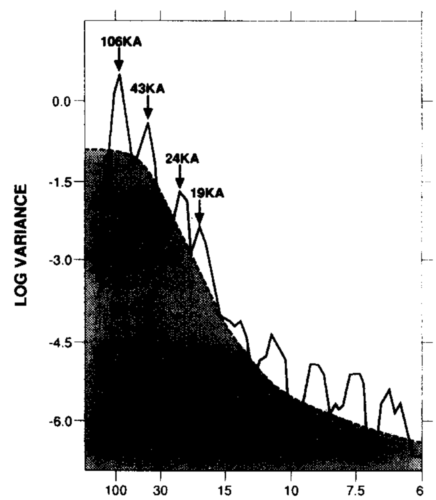

Fig. 2. Power spectrum of the ice volume based on deep-sea cores; KA = 1000 years; broken line indicates spectrum of red-noise-like background.

ear oscillators, representing to a greater or less extent the assumed essential physics. The common feature of such models is the assumption that ocean–atmosphere coupling (perhaps enriched by variation of ice extent) is a crucial physical ingredient, and that the annual cycle of solar heating, modified as necessary to include effects of orbital variations, is an important driving mechanism.

An excellent overview of coupled ocean–atmosphere models of El Niño and the so-called southern oscillation has appeared recently [4] whilst similar reviews of climatic models are contained, e.g., in refs. [5,6].

In this paper we examine two models, each consisting of coupled nonlinear oscillators, which represent coupled ocean–atmosphere dynamics in a simple way on appropriate timescales. In the first, based on the model of Saltzman et al. [7] of coupling between sea-ice extent and ocean temperature, we have added to the existing astronomical forcing an extra periodic forcing compatible with the earth's orbital variations. We have also considered an extra forcing in the form of band limited white noise. In the second, based on a model by Vallis [8] coupling local temperatures in the East and West Pacific with a current strength generated in part by these temperature differences, we have added to the annual forcing a second stochastic forcing compatible with the 20–60 day period of forced atmospheric Kelvin modes.

In each case, numerical experiments have indicated the replacement of chaotic behaviour in the singly periodically forced problem by "noisy periodicity" in the "quasi-periodically" forced problem. Here we use "noisy periodicity" in the sense introduced by Lorenz [9] to describe the temperature trace associated with the passage of "regular" waves in rotating heated annulus experiments (see, e.g., ref. [10]). The mean period of response is surprisingly different from either of the forcing periods, and is in the second case qualitatively similar to that of the El Niño events.

2. Long-term climatic variability: the Saltzman et al. oscillator

Long-term climatic variation, characterised most dramatically by the sequence of ice ages during the last million or so years, has been widely attributed to self-sustained oscillations arising from coupling between sea-ice extent and mean ocean temperature. A simple model proposed by Saltzman et al. [7] has received much attention, and Nicolis [11,12] has shown that the addition of periodic forcing to the model of Saltzman et al., intended to simulate variations in radiative heat input associated with variations in the earth's orbit (on timescales of $O(10^5)$ years), can produce a chaotic response.

Following Nicolis, we can write the system in the succinct form

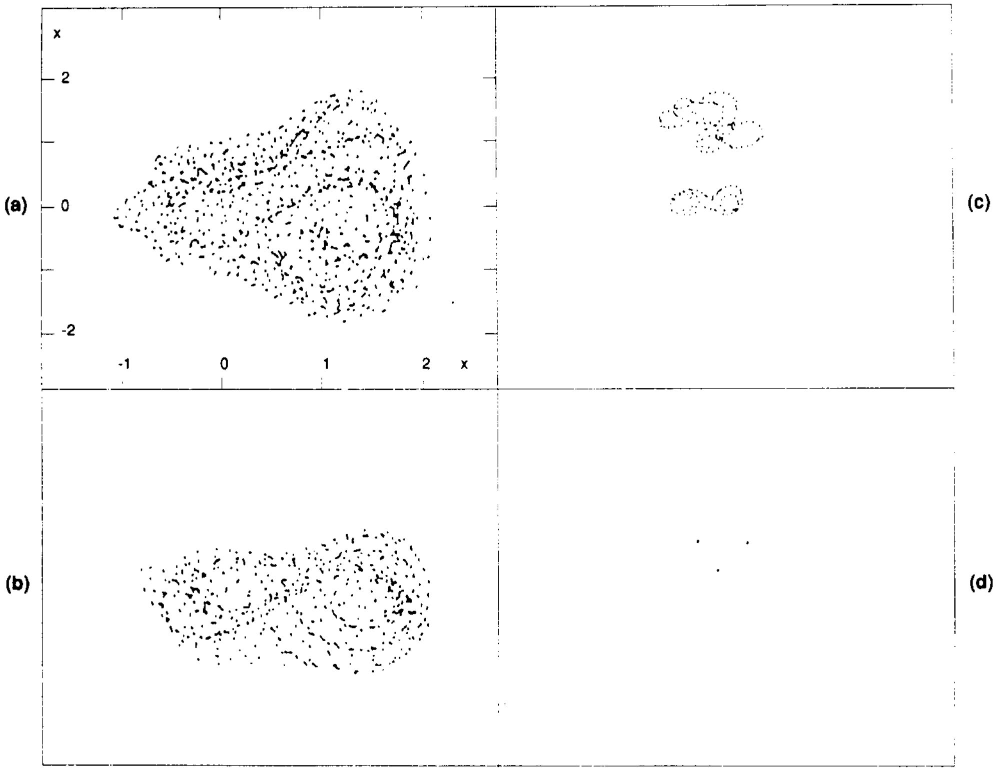

Fig. 3. (a) Poincaré map (interval $2\pi/\Omega$) for eqs. (2.1). Following Nicolis [5] we have chosen $\gamma_1=1.01$, $\gamma_2=1$, $\mu=0.1$, $q=0.74$, $\Omega=0.3$. (b) Poincaré map for eqs. (2.1) with an additional term $a \sin \omega t$, with $a=0.05$, $\omega=3\times10^4$; other parameter values as in (a). (c) Mean Poincaré map for eqs. (2.2). Here $\omega_1 \in [0.2, 0.4]$ and other parameter values are as for (b). (d) Mean Poincaré map for eqs. (2.2) with larger value of a ($=0.25$).

$$\dot{x}=y\,,$$
$$\dot{y}=\gamma_2 x-x^3+\mu(\gamma_1 y-x^2 y+q\sin\Omega t)\,, \qquad (2.1)$$

where x and $\mu x+y$ are respectively the scaled deviations of the size of sea-ice extent and of mean ocean temperature from reference states.

For varying γ_1, γ_2, μ, q and Ω, the response of this oscillator of course demonstrates the usual range of qualitative behaviour. For some values of the parameters of climatic significance, the behaviour is unmistakably chaotic, and fig. 3a comprises a Poincaré map for such a set of values. The addition to the forcing in (2.1) of a term $a\sin\omega t$, choosing $a=0.05$ and $\omega=30000$, to represent the weak annual forcing, has a dramatic effect on the Poincaré map, now reproduced as fig. 3b. The chaotic map is replaced by a map which corresponds to motion on a torus, indicating a quasi-periodic response by the oscillator. The qualitative character of the behaviour is highly sensitive to the value of a, and in fig. 4 we demonstrate this sensitivity over a range of values of a.

A second modification to (2.1) comprises the addition of band limited white noise to the basic single frequency forcing. Thus, by considering the equation

$$\ddot{x}-\mu(\gamma_1-x^2)\dot{x}-\gamma_2 x+x^3$$
$$=\mu q\sin\Omega t+a\sum_{i=1}^{100}\sin\omega_i t \qquad (2.2)$$

and taking $\omega_i\in[0.2, 0.4]$, we have, for the same values of parameters as in fig. 3b, a *mean* Poincaré map as indicated in fig. 3c. Note that we have defined the mean Poincaré map to be the set $\mathrm{M}\subset\mathbb{R}^2$:

$$\langle \mathrm{M}(\boldsymbol{x}(t_0))\rangle$$
$$=\{\langle x_1(t)\rangle, \langle x_2(t)\rangle\,|\,t=2\pi k/\Omega;\, k=1, 2, ...\}\,,$$

where $\boldsymbol{x}(t)$ are the realisations of the solutions of (2.2) for the initial condition $\boldsymbol{x}(t_0)$, and $\langle\ \rangle$ indicates an ensemble average obtained from a number of realisations arising from different ω_i [13].

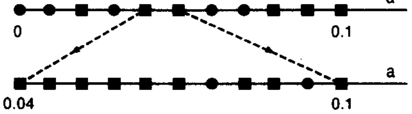

Fig. 4. Qualitative change in behaviour of solutions to (2.1) as a varies; results of computations at intervals of 0.01 between 0 and 0.1, and at intervals of 0.001 between 0.04 and 0.05; other parameter values as in fig. 3b. (●) Chaotic behaviour; (■) regular behaviour.

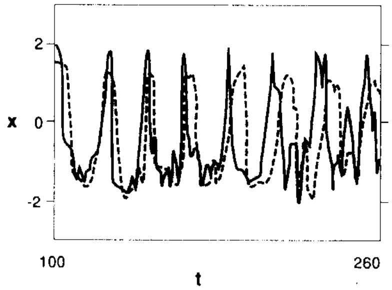

Fig. 5. Comparison of chaotic (solid line) and regular (dotted line) stochasticity, as described in text, for the interval 100 KA $<t<$ 260 KA.

When a is increased, we find that the mean Poincaré map takes the form of fig. 3d, corresponding to noisy periodicity; chaotic behaviour has been replaced by regular stochasticity. (Here we use the terms chaotic and regular stochasticity in conformity with the definition that a random process in which the most probable value of the maximum Lyapunov exponent is positive (non-positive) is called a chaotic (regular) stochastic process [13]. Actual signals may be difficult to identify correctly (fig. 5).)

In summary, we see that the chaotic behaviour of eq. (2.1) is readily destroyed by the addition of further weak forcing or coupling, and that the dependence on "perturbation strength" is extremely sensitive.

3. El Niño southern oscillation: the Vallis model

The El Niño southern oscillation (ENSO) is the predominant interannual variability of a tropical ocean–atmosphere system. It results in considerable fluctuations in rainfall, sea surface temperature (SST) and the intensity of the trade winds over the Pacific Ocean. Its range of variation and essential unpredictability impose considerable stress on ecological systems, and its effects on global weather and climate are accepted but poorly understood.

Many "simple" models have been designed to represent ocean–atmosphere interactions which may be involved in ENSO dynamics, and the very comprehensive survey [4] admirably summarises present thinking. We consider here a very simple two-point model of ENSO proposed by Vallis [8] which is based essentially on the East–West temperature advection equation and the equation of continuity. The model comprises three ordinary nonlinear equations describing the evolution of ocean current and the ocean mixed-layer temperatures at Eastern and Western stations in the Pacific, which may be written

$$du/dt=B(T_E-T_W)/2\Delta x-C(u-u^*)\,,$$
$$dT_W/dt=u(\bar{T}-T_E)/2\Delta x-A(T_W-T^*)\,,$$
$$dT_E/dt=u(T_W-\bar{T})/2\Delta x-A(T_E-T^*)\,, \quad (3.1)$$

where $\bar{T}$ is the constant temperature of deep ocean, A, B and C are constants, the term $B(T_E-T_W)/2\Delta x+Cu^*$ represents wind-produced stress, $-Cu$ represents mechanical damping, T^* is the temperature to which the ocean would relax in the absence of motion. The independent variable t is considered to vary on timescales of the ENSO phenomena, i.e., 2–4 years. Equations (3.1) constitute the basic model.

In our numerical experiments we have used the Vallis values, namely $A=1$ year^{-1}, $C=0.25$ month^{-1}, $B=2$ m^2 s^{-2} C^{-1}, $u^*=-0.45$ m s^{-1}, $T^*=12\,°$C and $\bar{T}=0\,°$C. These values characterise the background state and may have an important impact on the dynamics.

Vallis considered both the "free" behaviour of this system, and the effect on it of an annual forcing, and concluded that behaviour reminiscent of El Niño was possible. Our conjecture here is that the effects of the so-called Madden–Julian oscillations (MJO) of the atmosphere, associated with propagating and reflecting Kelvin waves, are also important in ENSO dynamics, as has been suggested by recent observational evidence [14]. For a detailed discussion see ref. [15].

We thus introduce a forcing in the ocean current equation consisting of (1) a component of constant amplitude frequency to represent the annual forcing and (2) a term intended to represent the MJO effects in the form of an atmospheric wind stress. We have chosen various forms for this forcing varying from pure white noise to sharply peaked distributions, with randomly chosen frequency (in the 40–60 day period range) and phase.

Some results are shown for the case in which we have taken

$$u^*=-0.45+0.1\cos\omega_1 t+\sum_{i=1}^{100}\delta_i\cos(\omega_i t+\phi_i)\,,$$

where δ_i, ω_i, ϕ_i are random variables such that

$$\delta_i\in(0.05, 0.3)\,,\quad 2\pi/\omega_i\in(20, 60)\text{ days}\,,$$
$$\phi_i\in(0, 2\pi)\,,\quad 2\pi/\omega_1=365\text{ days}\,.$$

Other parameters have been set at values found by Vallis to give a chaotic response. Figure 6 shows the variation of u, $\Delta T=T_E-T_W$ over a period of 100 years after 1000 years of integration, and fig. 7 shows two shorter intervals. Roughly periodic large anomalies are very prominent in all plots. For comparison we include (fig. 8) results for the case of simple periodic forcing (i.e. $\delta_i=0$ for all i) computed for identical parameter values.

Gross features are summarised in the spectrum (fig. 9), which shows a strong noisy peak corresponding to a mean period of $3\frac{1}{2}$–4 years in contrast to the spectrum (fig. 10) for the simply periodically forced case. All these features are reminiscent of El Niño and are discussed fully elsewhere [15]; the main point we wish to stress here is the appearance of a noisy periodicity in this "quasi-periodically" forced system at parameter values where a simple forcing gives chaotic behaviour.

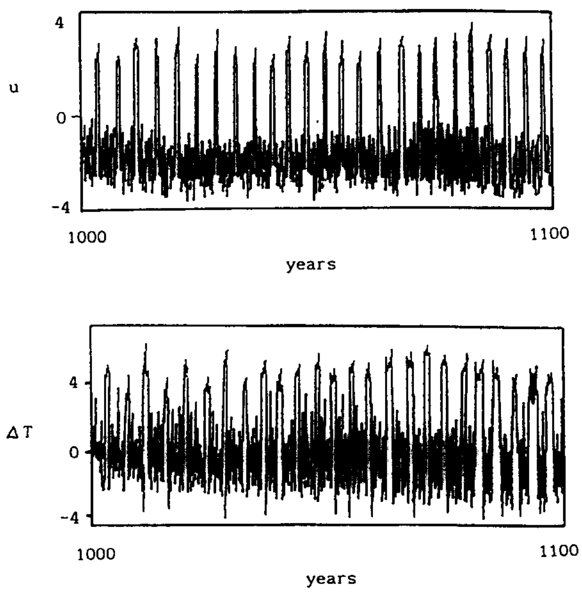

Fig. 6. Variation of u and ΔT over a period of 100 years as a result of integrating eqs. (3.1); values of parameters as specified in text, $\Delta T=T_E-T_W$.

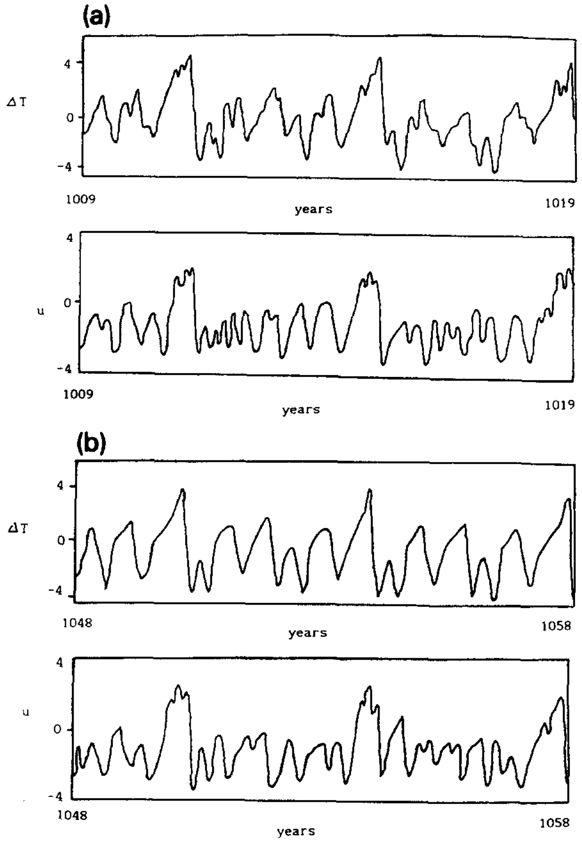

Fig. 7. Variations of u and ΔT over two short time intervals as indicated.

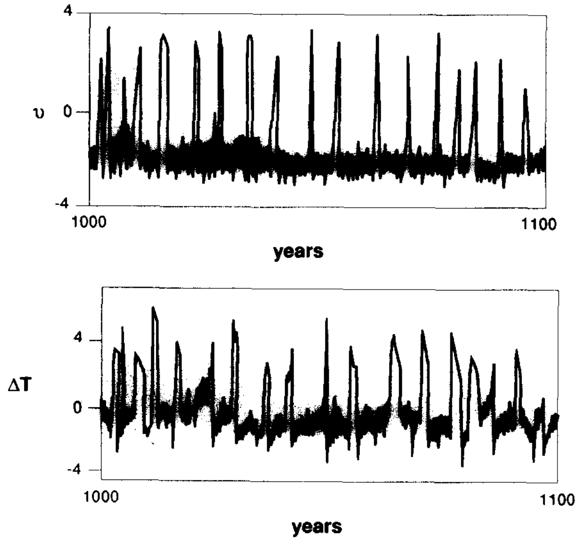

Fig. 8. Variations of u and ΔT over a period of 100 years as a result of integrating eqs. (3.1) for $\delta_i = 0$; other parameter values as specified in text, $\Delta T = T_E - T_W$.

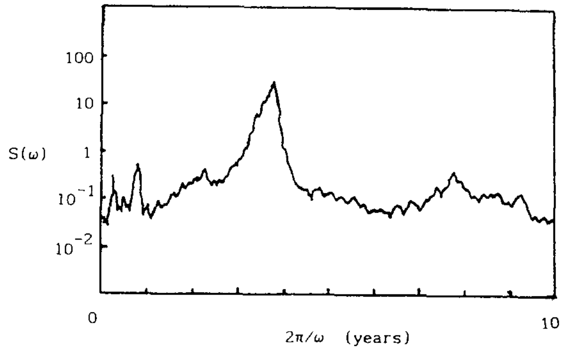

Fig. 9. Spectral density distribution for ΔT over the 100 years of fig. 6.

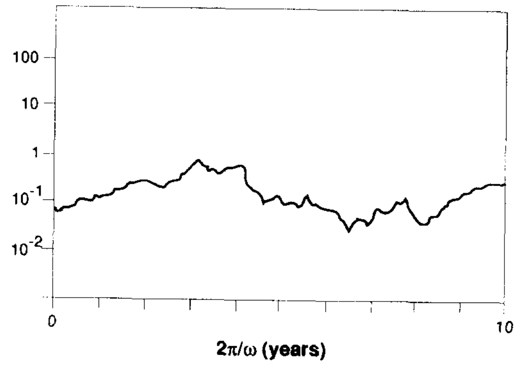

Fig. 10. Spectral density distribution for ΔT over the 100 years of fig. 8.

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