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ANALYTICAL METHOD OF CONTROLLING CHAOS IN DUFFING'S OSCILLATOR

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The presence of chaos both in nature and in man-made devices is very common and has been extensively demonstrated in the past decade. Very occasionally chaos is a beneficial feature, as in some chemical or heat and mass transport problems. However, in many other situations chaos is an undesirable phenomenon leading to oscillations, irregular operations, etc. Also, chaotic behaviour may be detrimental to the operation of various devices, as it cannot be predicted in detail.

Recently, Ott, Grebogi and Yorke [1, 2] showed that for a chaotic attractor one can obtain a desired attracting time-periodic motion by making only small time-dependent perturbations in an accessible system parameter. This method of controlling chaos is based on the observation that a chaotic attractor typically has embedded within it an infinite number of unstable periodic orbits. First, some of these unstable periodic orbits are determined, and then one which yields improved system performance is stabilized by adding small time-dependent perturbations to one of the system parameters. This method is very general and can be used for both low- and high-dimensional systems [3]. Recently, it was successfully applied to controlling an experimental chaotic system [4]. In addition to this method, the problem of controlling chaos has been mentioned in references [5, 6].

In what follows a different method is presented for controlling the chaotic behaviour of Duffing's equation,

$$\ddot{x} + a\dot{x} + bx + cx^3 = B_0 + B_1 \cos\Omega t, \tag{1}$$

where a, b, c, B_0, B_1 and Ω are constants.

It is well known that equation (1) shows chaotic behaviour for certain values of the parameters [7–10]. In many cases it can be shown that chaotic behaviour is obtained via a period-doubling bifurcation [8–10]. Recently, there have been some attempts to create an analytical criterion which allows one to estimate the chaotic domain in the parameter space [9–14]. Boundaries of the chaotic zone have been obtained by using the classical approximate theory of non-linear oscillations, in particular by examining approximate periodic solutions and studying particular types of higher order instabilities which precede the destruction of a periodic attractor in the variational Hill's type equation [15]. Here a similar procedure (based on the harmonic balance method) for controlling chaotic behaviour is to be presented.

First consider the first approximate solution in the form

$$x(t) = C_0 + C_1 \cos(\Omega t + \zeta_1), \tag{2}$$

where C_0, C_1 and ζ are constants. By substituting equation (2) into equation (1) it is possible to determine these constants [10–12]. To study the stability of the solution (2) a small variational term $\delta x(t)$ is added to equation (2), to give

$$x(t) = C_0 + C_1 \cos(\Omega t + \zeta_1) + \delta x(t). \tag{3}$$

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After some algebraic manipulations, a linearized equation with periodic coefficients for $\delta x(t)$ is obtained,

$$\delta\ddot{x} - a\delta\dot{x} + \delta x[\lambda_0 + \lambda_1 \cos\Theta + \lambda_2 \cos 2\Theta] = 0, \tag{4}$$

where $\lambda_0 = 3C_0^2 + (3/2)C_1^2$, $\lambda_1 = 6C_0C_1$, $\lambda_2 = (3/2)C_1^2$ and $\Theta = \Omega t + \zeta_1$. In the derivation of equation (4), for simplicity, it was assumed without loss of generality that $b = 0$. As one has a parametric term of frequency $\Omega - \lambda_1 \cos\Theta$, the lowest order unstable region is that which occurs close to $\Omega/2 \approx \sqrt{\lambda_0}$, and at its boundary one has the solution

$$\delta x = b_{1/2} \cos((\Omega/2)t + \rho). \tag{5}$$

To determine the boundaries of the unstable region one inserts equation (5) into equation (4), and the conditions for a non-zero solution for $b_{1/2}$ leads to the following criterion to be satisfied at the boundary:

$$(\lambda_0 - \Omega^2/4)^2 + a^2\Omega^2/4 - \lambda_1^2/4 = 0. \tag{6}$$

From equation (6) one obtains the interval $(\Omega_1^{(2)}, \Omega_2^{(2)})$ within which period-two solutions exist. Further analysis shows that at Ω_2 one has a stable period-doubling bifurcation for decreasing Ω and at Ω_1 an unstable period-doubling bifurcation for increasing Ω [12]. In this interval one can consider the period-two solution of the form

$$x(t) = A_0 + A_{1/2} \cos((\Omega/2)t + \rho) + A_1 \cos\Omega t, \tag{7}$$

where A_0, $A_{1/2}$, A_1 and ρ are constants to be determined. Again, to study the stability of the period-two solution one has to consider a small variational term $\delta x(t)$ added to equation (7). The linearized equation for $\delta x(t)$ then has the form

$$\delta\ddot{x} + a\delta\dot{x} + \delta x[\lambda_0^{(2)} + \lambda_{1/2c} \cos(\Omega/2)t + \lambda_{1/2s} \sin(\Omega/2)t + \lambda_{3/2} \cos((3\Omega/2)t + \rho)$$
$$+ \lambda_{1c}^{(2)} \cos\Omega t + \lambda_{1s}^{(2)} \sin\ + \lambda_2^{(2)} \cos 2\Omega t] = 0, \tag{8}$$

where

$$\lambda_0^{(2)} = 3(A_0^2 + 0{\cdot}5A_{1/2}^2 + 0{\cdot}5A_1^2), \qquad \lambda_{1/2c} = 3A_{1/2}(2A_0 + A_1)\cos\rho,$$
$$\lambda_{1/2s} = 3A_{1/2}(A_1 - 2A_0)\sin\rho, \qquad \lambda_{3/2} = 3A_1A_{1/2}, \qquad \lambda_{1c}^{(2)} = 6A_0A_1 + (3/2)A_{1/2c}^2 \cos 2\rho,$$
$$\lambda_{1s}^{(2)} = -(3/2)A_{1/2s}^2 \sin 2\rho, \qquad \lambda_2^{(2)} = (3/2)A_1^2.$$

The form of equation (8) enables one to find the range of existence of a period-four solution, represented by

$$\delta x = b_{1/4} \cos((\Omega/4)t + \rho) + b_{3/4} \cos((3\Omega/4)t + \rho). \tag{9}$$

After inserting equation (9) into equation (8) the condition for non-zero solutions for $b_{1/4}$ and $b_{3/4}$ yields the following set of non-linear algebraic equations for Ω, $\cos\phi$ and $\sin\phi$ to be satisfied for the solutions existence:

$$(\lambda_{1/2s} + \lambda_{1s}^{(2)}) - 0{\cdot}5(\lambda_{1/2c} - \lambda_{1c}^{(2)})(-a\Omega/2 + \lambda_{1/2s} - \lambda_{3/2}\sin\rho) = 0,$$
$$((9/8)\Omega^2 + 0{\cdot}5\lambda_0^{(2)} + \lambda_{3/2}\cos\rho) - 0{\cdot}5(\lambda_{1/2c} + \lambda_{1c}^{(2)})(\lambda_{1s}^{(2)} + \lambda_{1/2c}) = 0,$$
$$(-(3/2)a\Omega - \lambda_{3/2}\sin\rho) - 0{\cdot}5(\lambda_{1/2c} + \lambda_{1c}^{(2)})(\lambda_{1s}^{(2)} + \lambda_{1/2c}) = 0. \tag{10}$$

By solving equations (10) by a numerical procedure it is possible to obtain $\Omega_1^{(4)}$ and $\Omega_2^{(4)}$, the frequencies of the stable and unstable period-four bifurcations.

Now it is assumed that the Feigenbaum model [16] of period doubling is valid for the system: i.e.,

$$(\Omega_{1,2}^{(2^n)} - \Omega_{1,2}^{(2^{n-1})})/(\Omega_{1,2}^{(2^{n+1})} - \Omega_{1,2}^{(2^n)}) \underset{n\to\infty}{\to} \delta, \tag{11}$$

where $\delta = 4{\cdot}669\ldots$ is the universal Feigenbaum constant and $n = 1, 2, \ldots$ Although it has not been proved that all period-doubling bifurcations occur via the Feigenbaum model, there are examples where this model can be taken as a good approximation to the real phenomena [17, 18].

To obtain approximate values of the limits of period-doubling bifurcations (accumulation points) one can replace the limit in expression (11) by an equality. After that, one can define $\Delta\Omega_{1,2}^{(2^n)} = \Omega_{1,2}^{(2^n)} - \Omega_{1,2}^{(2^{n-1})}$. Then it is easy to show that $\Delta\Omega_{1,2}^{(2^n)}, \Delta\Omega_{1,2}^{(2^{n+1})}, \ldots$ form infinite geometrical series with a ratio $1/\delta$. With both stable and unstable period-two and period-four boundaries, by using this approximation, one obtains

$$\Omega_1^{(\infty)} = \Omega_1^{(2)} + \Delta\Omega_1/(1 - 1/\delta), \qquad \Omega_2^{(\infty)} = \Omega_2^{(2)} - \Delta\Omega_2/(1 - 1/\delta),$$

where $\Delta\Omega_1 = \Omega_1^{(4)} - \Omega_1^{(2)}$ and $\Delta\Omega_2 = \Omega_2^{(2)} - \Omega_2^{(4)}$.

The domain in which chaotic behaviour can occur is supposed to be between the limits of unstable and stable period-doubling cascades, in the interval $(\Omega_1^{(\infty)}, \Omega_2^{(\infty)})$, and of course to expect chaos one must have

$$\Omega_1^{(\infty)} < \Omega_2^{(\infty)}. \tag{12}$$

More details about this method can be found in references [13, 14]. With this approach, besides estimating the chaotic regions in parameter space, one can evaluate analytically the approximate unstable orbits, or at least the regions in parameter space where they exist. The analysis can be used to control equation (1) by changing the parameter Ω in the range $\Omega \in [\Omega - \Omega^*, \Omega + \Omega^*]$. It should be noted here that the frequency Ω is the parameter which can be very easily changed in real experimental systems modelled by equation (1). From Figure 1 one can find that for $\Omega^* < 0{\cdot}12$ one can obtain different types of periodic behaviour; from period-one to theoretically period-2^n ($n = 1, 2, \ldots$). Higher order periodic orbits with $n > 4$ are difficult to obtain, as the Ω intervals of the existence of these solutions are very small. An example of the control of a few of the periodic orbits is shown in Figure 2. The co-ordinate x of the Poincaré map is plotted as a function of the discrete time $\hat{t} = 2\pi n/(\Omega \pm \Omega^*)$, $n = 1, 2, \ldots$ The frquency perturbations were programmed to control successively four different periodic orbits. The times at which the control was switched from stabilizing one periodic orbit to stabilize another are labelled by arrows in the figure.

Recently, it has been shown that addition of a small quasi-periodic noise of the form

$$\eta(t) = \sum_{i=1}^{N} A_i \cos(v_i t + \phi_i), \tag{13}$$

where $A_i \ll B_{0,1}$ are constant, and v_i and ϕ_i are time independent random variables, shifts the period-doubling bifurcation point, and decreases the size of the zone of existence of each period-2^n solution, and can even eliminate chaos [14, 19].

This phenomenon shows that one can reduce the range in which the control parameter is allowed to vary by adding simultaneously noise into the system: i.e.,

$$\ddot{x} + a\dot{x} + bx + cx^3 = B_0 + B_1 \cos((\Omega \pm \Omega^*)t) + \sum_{i=1}^{N} A_i \cos(v_i t + \phi_i). \tag{14}$$

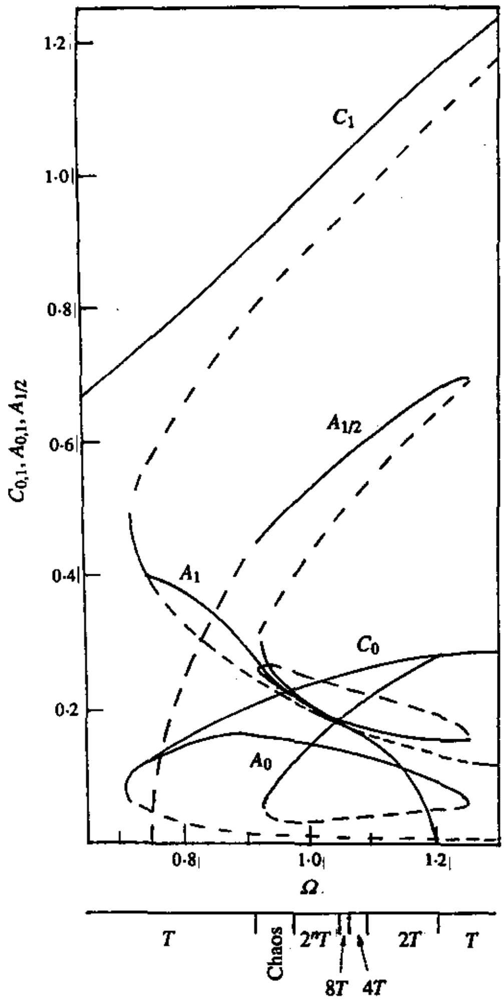

Figure 1. Parameters of period-one and period-two approximate solutions (2) and (7), and the Ω intervals of existence of period-four and period-eight solutions; $a=0{\cdot}05$, $b=0$, $c=1$, $B_0=0{\cdot}03$, $B_1=0{\cdot}16$. The solid line indicates stable solutions while the broken line indicates unstable solutions.

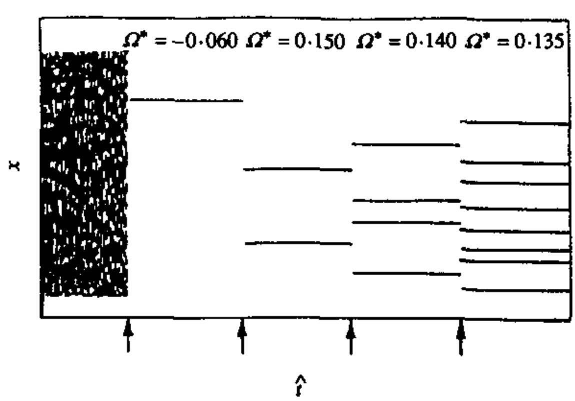

Figure 2. Successive control of period-one, -two, -four and -eight orbits. The arrows indicate the times of switching; $a=0{\cdot}05$, $b=0$, $c=1$, $B_0=0{\cdot}03$, $B_1=0{\cdot}16$, $\Omega=0{\cdot}97$.

Quasi-periodic noise given by equation (13) is an approximation of the realization of the band-limited white noise stochastic process with zero mean and a spectral density

$$s(\nu)=\begin{cases}\sigma^2/(\nu_{max}-\nu_{min}), & \nu\in[\nu_{min},\nu_{max}]\\ 0, & \nu\notin[\nu_{min},\nu_{max}]\end{cases}, \qquad (15)$$

where σ^2 is the intensity of the noise and $[\nu_{min}, \nu_{max}]$ is the interval of the frequencies considered [14, 19]; this approximate realization can be easily simulated experimentally.

An example of this type of control is shown in Figure 3. In this case, to illustrate the effect of the control, solutions of appropriate periods as perturbed by the noise expression (13) were obtained and are plotted in the figure.

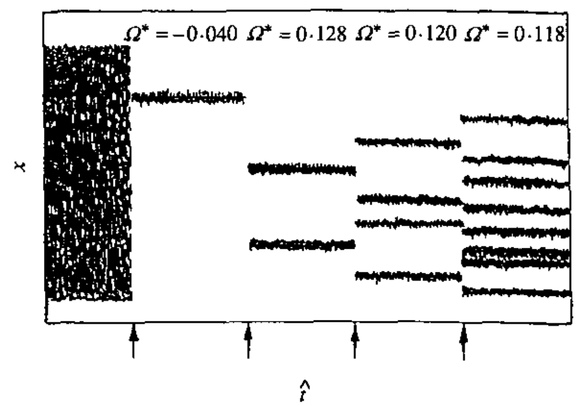

Figure 3. Successive control of period-one, -two, -four and -eight orbits with noise added to the system. The arrows indicate the times of switching; $a=0{\cdot}05$, $b=0$, $c=1$, $B_0=0{\cdot}03$, $B_1=0{\cdot}16$, $\Omega=0{\cdot}97$, $A_i=0{\cdot}004$, $\nu_{min}=0{\cdot}9$, $\nu_{max}=1{\cdot}1$, $N=200$.

The analytical technique for controlling chaos in Duffing's equation, as presented in this paper, is based on (a) the approximate period-one, -two and -four solutions and their stability limits computed by the harmonic balance method, and (b) Feigenbaum's universal constant for the asymptotic ratio of the stability intervals of the 2^n and 2^{n+1} periodic solutions. It can be applied to the class of Duffing's equations for which the harmonic balance method analysis shows the possibility of period-doubling bifurcation ($\lambda_1=0$ in equation (4)).

An interesting novelty of this method is the use of noise to reduce the necessary range over which the control parameter is varied. This is important as the method requires larger perturbations of the control parameter than the method of references [1, 2].

The method presented is less general than the Ott–Grebogi–Yorke method. On the other hand, it has some advantages. The control can be applied at any time and one can switch from one periodic orbit to another without returning back to the chaotic behaviour, although after each switch transient chaos is observed. The lifetime τ of this transient chaos strongly depends on initial conditions, and here it has been found that $\tau<400(\pi/(\Omega\pm\Omega^*))$. In the method one does not have to wait until the trajectory is close to an appropriate unstable orbit. In some cases this time can be long. All the information necessary for the control can be obtained analytically from equation (1). One does not need to have long-time numerical solutions (time series). It is not necessary to consider any information obtained from a Poincaré cross-section. This allows one to take the frequency Ω as a control parameter without worrying that by varying Ω one is changing the Poincaré cross-section.

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