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OSCILLATIONS OF A QUASI-PERIODICIALLY FORCED SYSTEM WITH DRY FRICTION

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Recently, there has been a growing interest in studying forced dynamical systems with multiple incommensurate forcing terms. Forcing at (at least) two irrationally related frequencies is common in engineering systems; indeed, forcing at a single frequency is likely to be the exception rather than the rule. *A fortiori,* in naturally occurring dynamical systems, physical or biological, a multi-peaked spectrum of forcing is to be expected.

The dynamics of these systems is generally substantially different from that of single frequency appropriate counterparts: for example, the types of invariant sets supporting such systems are more complicated. Quasi-periodic forcing implies that the most elementary invariant sets are tori, rather than periodic orbits as in single frequency driving. In addition, new invariant sets can be found, as in the case of recently discovered strange non-chaotic attractors [1–10].

Works on quasi-periodically forced system have been written from mathematical [1, 11], physical [2–10] and, more recently, from mechanical engineering [9, 10] standpoints.

In our previous work [8] we found some interesting properties of frequencies spectrum of quasi-periodically forced systems, which can be summarized in Figure 1. Let us say that we have a system with forcing frequencies ω_1 and ω_2. Consider ω_1 as a constant and take ω_2 as a control parameter. When ω_1/ω_2 is rational the system considered is of course periodically forced, and if it is not chaotic we observe a relatively simple frequency spectrum. Going with ω_2 out of the point $\omega_1/\omega_2 = \textit{rational number}$ we observe a trifurcation of each frequency in the power spectrum: i.e., in a vicinity of each frequency component two new symmetrical component arise. A further increase or decrease of ω_2 results in the origin of other frequency components in the mechanism described.

In this letter we show that such similar properties of frequency spectra are characteristic of experimental quasi-periodically forced systems. We propose a simple method for predicting frequency spectrum components.

The experimental system under investigation here is the one presented in Figure 2. In this test rig oscillating bodies 1 and 2 are in contact on their surfaces, through a dry friction force. A detailed description of our experiments has been given elsewhere [9, 10]. In reference [9] we showed that the friction force between both bodies can be chaotic and in reference [10] we estimated the dimensions of the possible attractors, pointing out the importance of the so-called normal degree of freedom. Here, we concentrate on the properties of frequency spectra of the displacements x_1 and x_2. Typical examples of these spectra are shown in Figures 3(a) and (b). In Figure 3(a) we have the spectra obtained when the ratio ω_1/ω_2 is very near to the rational value of 3. In this case the power spectra have significant peaks at ω_1, $2\omega_1$, $3\omega_1$ and $4\omega_1$, and of course the responses $x_1(t)$ and $x_2(t)$ are $2\pi/\omega_1$-periodic. In Figure 3(b) we present the spectra for $\omega_1/\omega_2 = 3{\cdot}102\ldots$, which is slightly higher than the value of the previous case. In Figure 3(b) is shown the existence of the other peaks which are present in the vicinity of the peaks of Figure 3(a). New peaks appear in pairs symmetrical in respect to the main ones: for example, at frequencies

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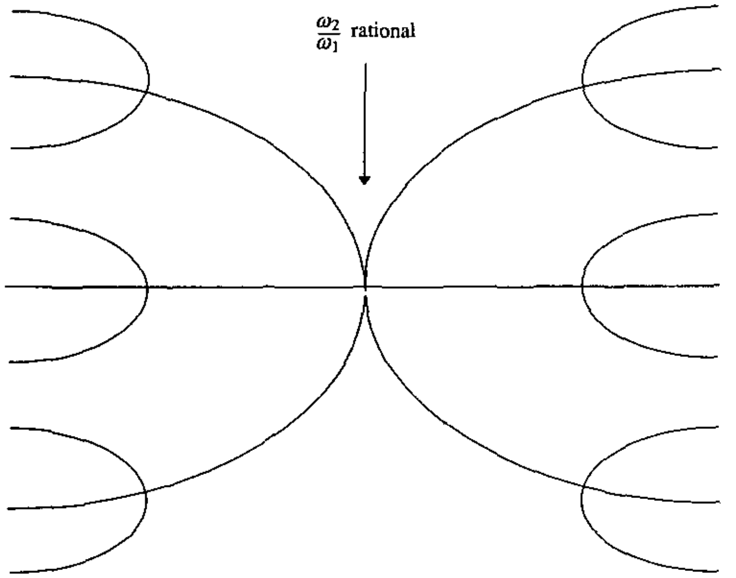

Figure 1. Typical structure of the power spectrum of quasi-periodically forced system.

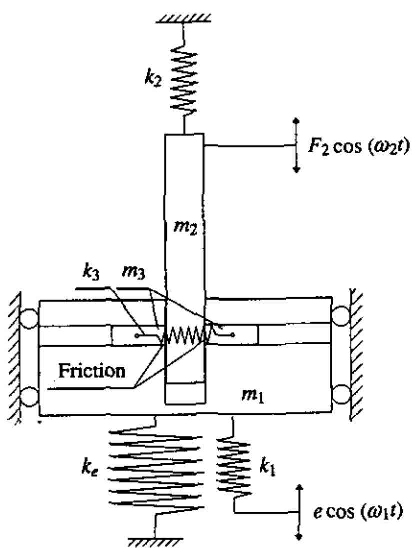

Figure 2. The experimental rig.

$(4\omega_1-\omega_2)$ and $(\omega_2-\omega_1)$, and $(7\omega_1-2\omega_2)$ and $(2\omega_2-5\omega_1)$—both marked in Figure 3(b). Generally, we observe a structure of the power spectra similar to that described in Figure 1, which is characteristic for the whole range of system parameters [12].

The described structure of the power spectra allows us to develop the following method for predicting components of the power spectra. This method is based on the following construction. We plot the diagram of the possible spectrum frequencies as a function of the excitation frequency ω_2 with the other frequency ω_1 fixed; see Figure 4. The diagram shows the lines describing the frequencies of the type $f_{ij}=\pm i\omega_1 \pm j\omega_2$. The level of complication of the picture depends on the maximum values of the numbers i and j selected. The

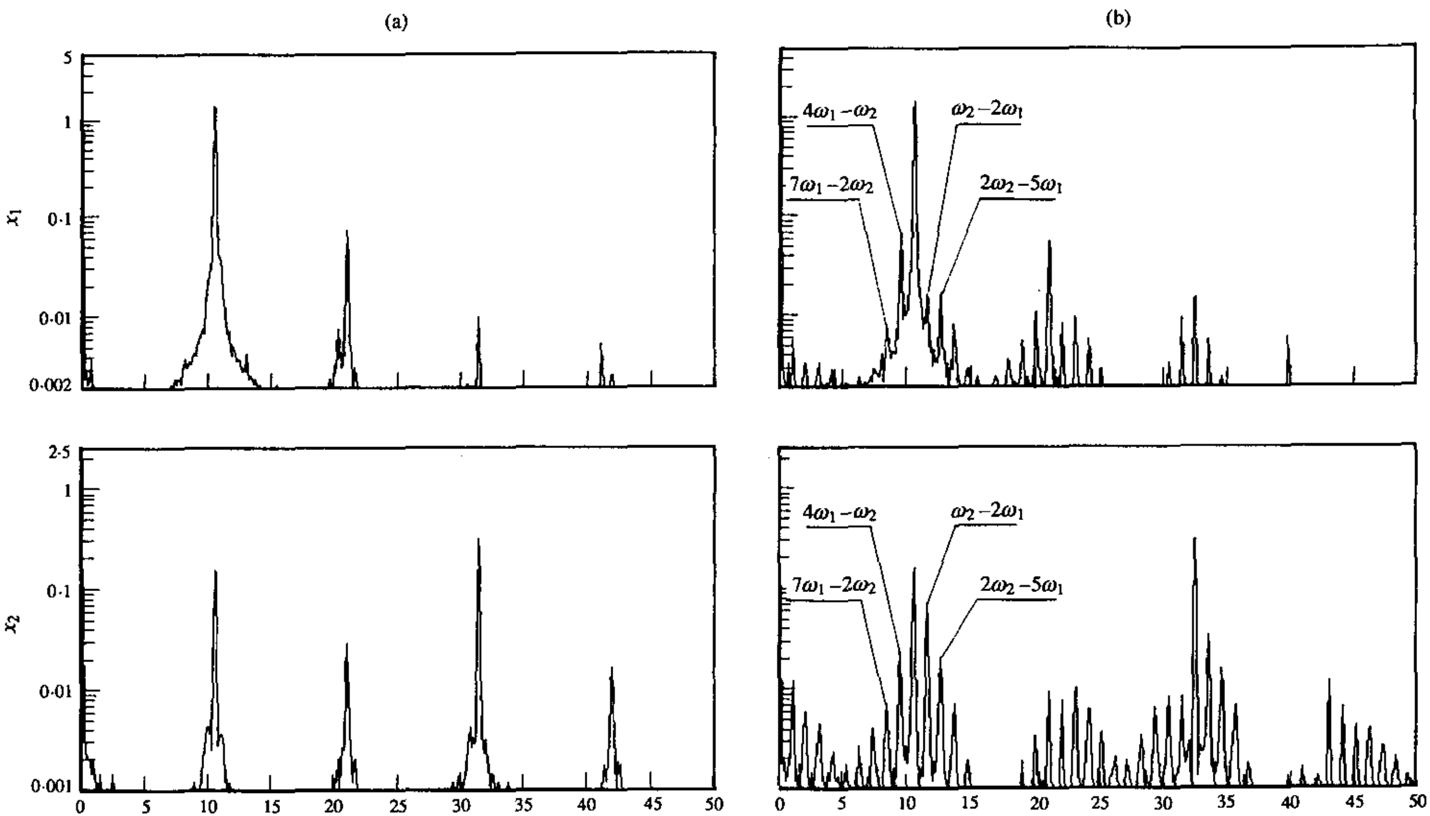

Figure 3. Examples of the power spectra of $x_1(t)$ and $x_2(t)$ (subsystem 1 the top part, subsystem 2 the bottom part). (a) $\omega_1=10{\cdot}45$, $\omega_2=31{\cdot}45$ Hz; (b) $\omega_1=10{\cdot}52$, $\omega_2=32{\cdot}62$ Hz.

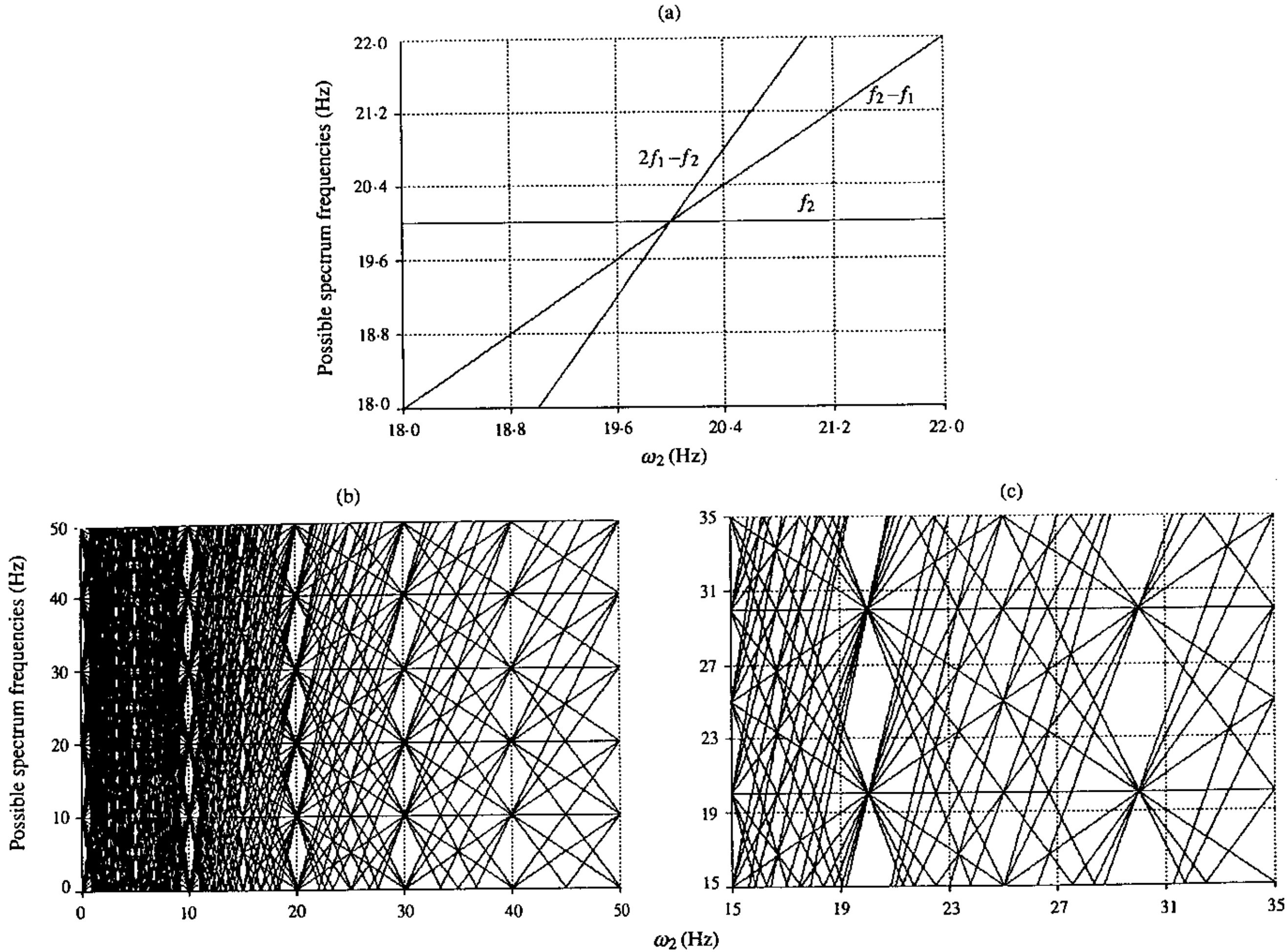

Figure 4. Construction of the prediction method ($\omega_1 = 10{\cdot}0$ Hz): (a) main idea; (b) plot for $i=j=21$; (c) enlarged part of the plot for $i=j=21$.

structure is repeatable; this means that by decreasing the window size similar patterns can be obtained, as shown in Figures 4(a) and 4(b). The number of i and j which is sufficient for the analysis of our experimental results is about 21.

Of course, not all of the predicted peaks are always present in the power spectra, but the accuracy of the prediction is high, as is shown in Figures 5(a) and 5(b), where we have compared the values of the predicted peaks with these found in experiments.

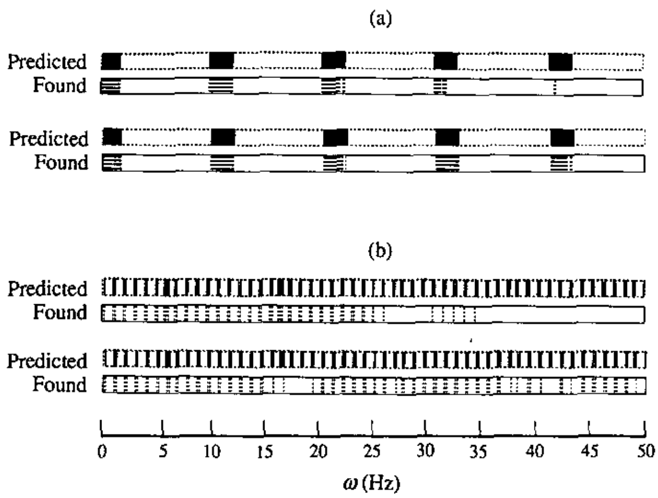

Figure 5. Predicted and observed components of the power spectra. (subsystem 1 the top part, subsystem 2 the bottom part). (a) $\omega_1 = 10{\cdot}45$, $\omega_2 = 31{\cdot}45$ Hz; (b) $\omega_1 = 10{\cdot}52$, $\omega_2 = 32{\cdot}62$ Hz.

To summarize, we have found that a specific structure of the power spectra of quasi-periodically forced systems which was previously determined in numerical experiments [8] is present in real experimental systems as well, and seems to be general. The existence of this structure allows one to develop a graphical method for prediction of the main freqencies in the power spectra. It can also be used to identify routes to chaos. Determination of the significant peak in the power spectrum, which is not predicted by our method, can be taken as an indicator of Hopf bifurcation. We have to note here that Hopf bifurcation in a two frequency forced system creates the third incommensurate frequency and a three frequency torus as an invariant set. It is well known that such a torus is very unstable [1], and that its break-up creates a strange attractor. A similar method which considers frequencies $(i\omega_1 + j\omega_2)/2^n$, $i, j, n = 1, 2, \ldots$, can be developed to identify the torus doubling route to chaos.

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