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Stochastic Resonance in Chaotically Forced Systems

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Abstract—The realization of stochastic resonance in chaotically forced systems is discussed and a new measure of it is introduced.

The phenomenon of stochastic resonance was introduced by Benzi *et al.* [1–3] to explain the periodicity of Earth's ice ages. They showed that the small periodic perturbations due to the Earth's wobble could lead to large-scale climatic changes via nonlinear cooperative effect between periodic and random fluctuations. Several theoretical and experimental analyses have appeared in which the main characteristics of this phenomenon were described [4–13]. The primary signature of stochastic resonance is that the addition of random noise in the periodically modulated systems can improve the signal-to-noise ratio as a signal peak in power spectra appears or becomes sharper, relative to that observed with no externally injected noise. This property can be summarized in Fig. 1, where the power spectra of the response of the dynamical system are presented. In both cases an external periodic excitation $\epsilon \cos \Omega t$ and small random noise $\zeta(t)$ are inputs into the dynamical system. When the intensity of random noise is small a power spectrum of the response $x(t)$ is nearly flat as shown in Fig. 1(a). In the second case—Fig. 1(b) random noise input $\zeta(t)$ is larger and a power spectrum has sharper peak and larger noise-to-signal ratio. Stochastic resonance is essentially a nonlinear phenomenon, requiring the presence of multiple stable

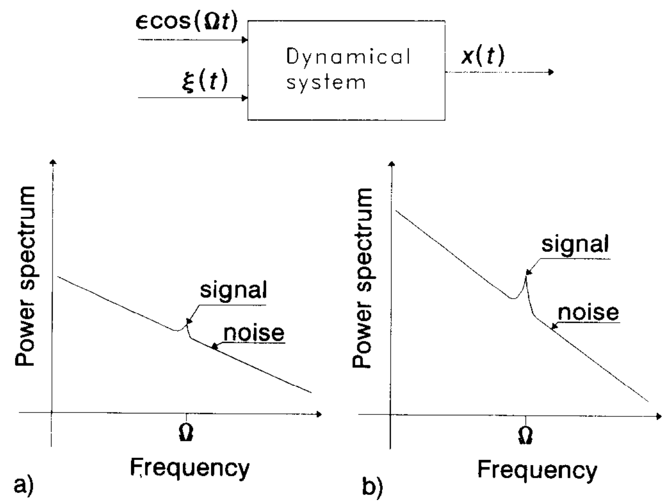


Fig. 1. A property of the phenomenon of stochastic resonance: (a) small noise; (b) larger noise.

states; the basic ingredients are generic enough that it is usually expected to occur in a wide variety of physical systems. Most of the papers on stochastic resonance phenomena [1–12] consider regular (not chaotic) systems. The only exception [13] describes stochastic resonance in the Chua's circuit in the regime of the two co-existing symmetrical Rossler type attractors. It was shown that in the presence of both sinusoidal and random forcing the co-existing attractors merge together giving birth to the double scroll attractor and result in the amplification of the sinusoidal signal intensity.

In this paper we investigate the possibility of stochastic phenomena in chaotically forced systems. Based on the autocorrelation function of the frequency spectra we introduced a new measure of stochastic resonance which is especially suitable for chaotically forced systems. Chaotic forcing has some advantages in comparison with periodic one and recently a growing interest in chaotically forced systems is observed, for example refs [14–17]. Using chaos we can for example correct certain nonlinear out-of-phase problems, eliminate fractal basin boundaries [16] and control unstable orbits [18]. Dynamics of chaotically forced systems is strictly connected with a phenomenon of synchronization of chaotic signals [14–16, 17, 19–24]. Synchronization in chaotic systems seems to be interesting not only from the theoretical point of view. It gives rise to new applications, such as using chaos for secure communication [23, 24].

In our numerical investigation we considered the dynamical behaviour of a particular yet representative case of a pair of unidirectionally coupled Duffing's oscillators:

$$\frac{\mathrm{d}x_1{}^2}{\mathrm{d}t^2} + \beta_1\frac{\mathrm{d}x_1}{\mathrm{d}t} - x_1 + x_1{}^3 = \gamma\cos(\Omega t) \tag{1a}$$

$$\frac{\mathrm{d}x_2{}^2}{\mathrm{d}t^2} + \beta_2\frac{\mathrm{d}x_2}{\mathrm{d}t} - x_2 + x_2{}^3 = \epsilon x_1 + \zeta(t) \tag{1b}$$

where β_1, β_2, γ, ϵ and Ω are constant. $\zeta(t)$ is a white noise with intensity D and the following properties

$$\langle\zeta(t)\rangle = 0 \qquad \langle\zeta(t)\zeta(t+\tau)\rangle = D\delta(\tau)$$

where $\langle\ \rangle$ and $\delta(\tau)$ denotes the averaging operator and the delta function, respectively. Equations (1) are typical bistable systems and in the absence of any forcing they are characterized by two stable fixed points x_- ($x = -1$, $\mathrm{d}x/\mathrm{d}t = 0$) and x_+ ($x = 1$, $\mathrm{d}x/\mathrm{d}t = 0$).

In our system (1) the output from the first oscillator excites the second one. We consider the case when the response of the first oscillator is chaotic, e.g. $\beta_1 = 0.15$, $\gamma = 0.3$, $\Omega = 1.0$ [25]. In the chaotic regime fixed points x_- and x_+ become unstable but chaotic response may show some regularity over short timescales with occasional substantial deviations, as the trajectory spends time in the neighbourhood of one of unstable fixed points (x_- or x_+) between sporadic excursions to the neighbourhood of another. The phase portrait and the power spectrum of the chaotic forcing x_1 are shown in Fig. 2(a) and (b). The parameters of the equation (1b) β_2 and ϵ are taken in such a way that in the absence of random noise ($D = 0$) two co-existing symmetrical chaotic attractors exist as shown in Fig. 3(a). In Fig. 3(b) we show a power spectrum of x_3. In this case the chaotic forcing is too small to direct a trajectory out of the neighbourhood of one of the unstable fixed points (x_- or x_+) to the neighbourhood of the other one. With a noise added to the system ($D \neq 0$) we observe a phenomenon of the stochastic resonance. It is observed as two co-existing attractors shown in Fig. 3(a) merge together and create one chaotic attractor shown in Fig. 4(a). In the power spectrum of x_3 shown in Fig. 4(b) the peaks are sharper than in the noiseless case and become even sharper with the increase of the noise intensity D as shown in Fig. 4(c). The physical mechanism of the observed stochastic resonance is the noise induced 'chaos–chaos' intermittency of Anishchenko [26]. It should be noted that the term

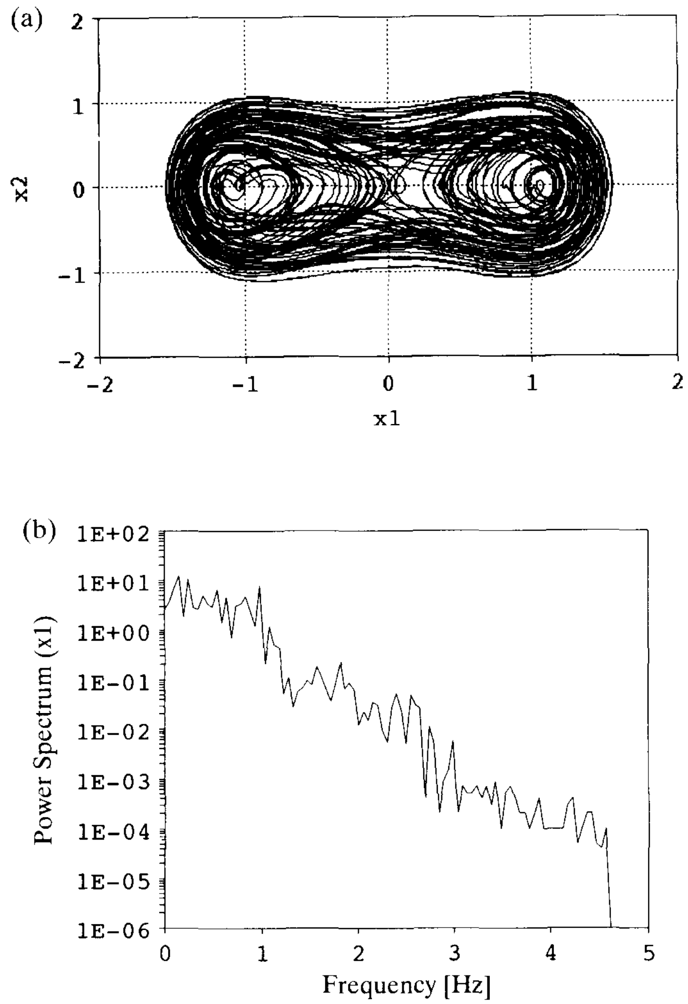
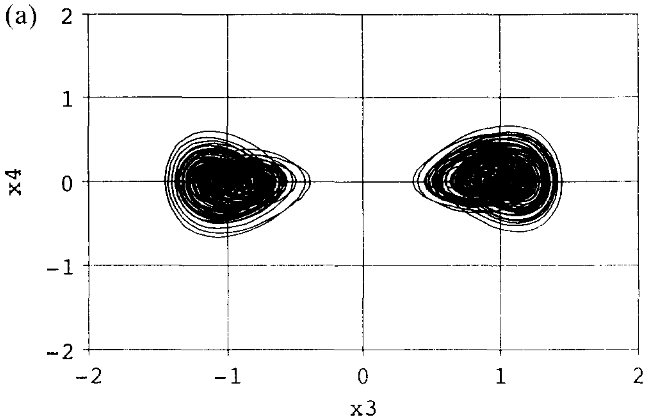


Fig. 2. The characteristics of the chaotic forcing x_1: $\beta_1 = 0.15$, $\gamma = 0.3$, $\Omega = 1.0$; (a) phase portrait; (b) power spectrum.

'intermittency' includes here the phenomena when two nearby attractors, of possible different types (e.g. limit cycles, torus, chaotic attractor) collided with each other at some critical parameter μ^* in the absence of noise. Such bifurcation transitions include the 'cycle–chaos' type intermittency, the 'torus–chaos' intermittency, and the 'chaos A–chaos B' intermittency like in the example considered in this paper.

As in the case of the chaotic forcing the power spectrum of signal x_1 is continuous [see Fig. 2(a)] it is impossible to define a signal-to-noise ratio measure of stochastic resonance like in the case of periodically forced systems as it is not clear which peak in the spectrum of x_3 has to be taken as a signal. To avoid this trouble we propose a new measure of stochastic resonance: the autocorrelation function of the power spectra, which is defined as

$$C = \frac{1}{M} \sum_{i=1}^{M} \frac{\sum_{j=1}^{M} P(j + i \bmod M) P(j)}{\sum_{j=1}^{M} P(j) P(j)}$$



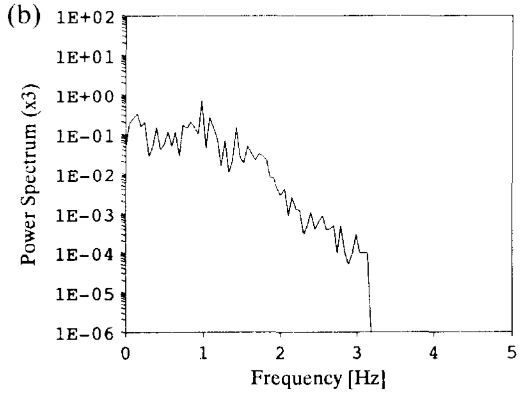


Fig. 3. The characteristics of the response x_3: $\beta_2 = 0.3$, $\epsilon = 0.25$, $D = 0$; (a) phase portrait; (b) power spectrum.

where $P(j)$ is the value of the power at the jth frequency index, and M is the number of discrete points in the spectrum. This provides a good measure of the 'flatness' of the spectrum, and C takes value 1 when the spectrum is completely flat and value 0 when there are δ peaks. Now we introduce a quantity which can serve as a sensitive indicator of the sharpness of the peaks. It is given by

$$S = -\log_{10} C$$

where $S = 0$ is a signature of the flat spectrum (no peaks), and $S = \infty$ is the signature of (very sharp) δ peaks. In Fig. 5 the value of S is plotted vs noise intensity D. It is clearly visible that the sharpness increases with increasing noise up to $D = 0.09$ which is typical for features of stochastic resonance phenomenon.

Finally, we remark that the stochastic resonance phenomenon has been numerically observed in system (1) by changing either the parameter β_2, or the parameter ϵ in the vicinity of the transition point from two co-existing chaotic attractors to one chaotic attractor and seems to be robust.

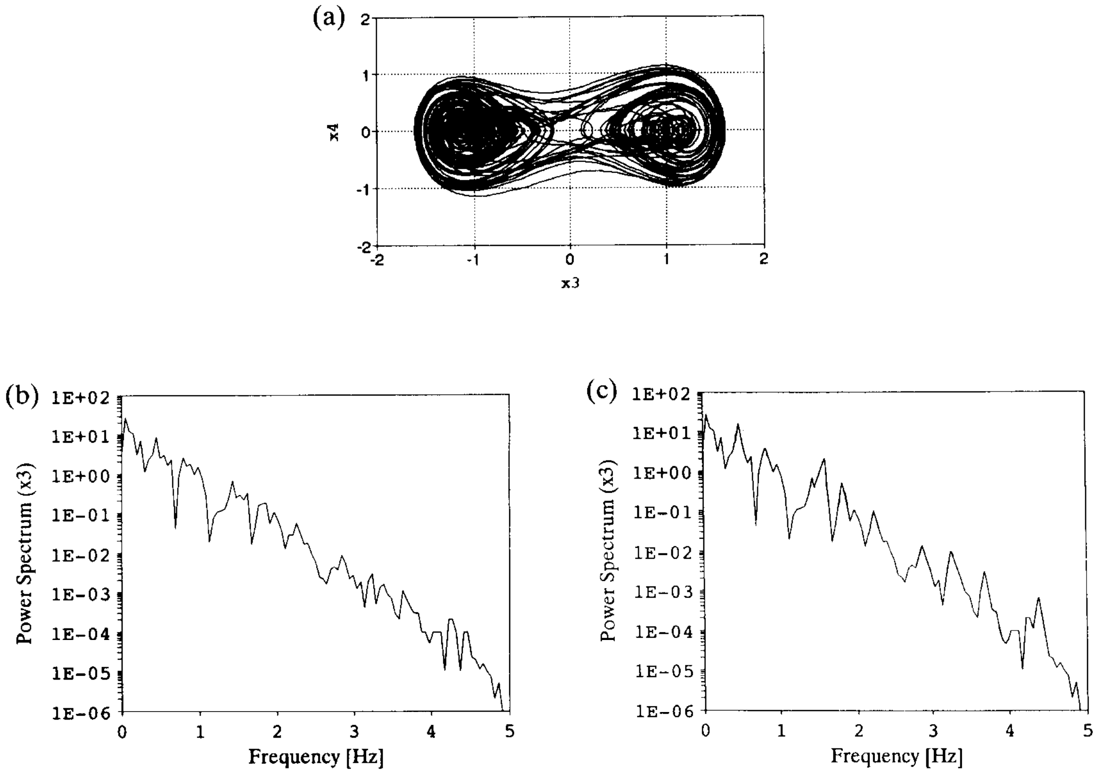


Fig. 4. The characteristics of the response x_3: $\beta_2 = 0.3$, $\epsilon = 0.25$; (a) phase portrait: $D = 0.02$; (b) power spectrum: $D = 0.02$; (c) power spectrum: $D = 0.04$.

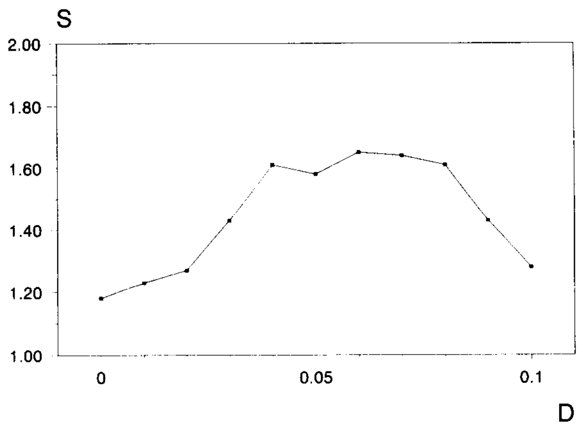


Fig. 5. The autocorrelation function of the spectrum as a measure of stochastic resonance.

To summarize we showed here that the stochastic resonance phenomenon can occur in bistable or multistable systems, not only in the presence of periodic forcing but in the case of chaotic forcing as well. For chaotically forced systems the autorcorrelation function of the power spectra can be considered as a measure of the stochastic resonance. The noise induced 'chaos–chaos' intermittency seems to be the typical mechanism of stochastic resonance in chaotic systems with both periodic and chaotic forcing.

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