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Complex Behaviour of a Quasiperiodically Forced Experimental System with Dry Friction

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Abstract—A mechanical experiment is described in which dry friction provides a nonlinear coupling between forced linear oscillators. Interpretation of the aperiodic behaviour of the system suggests that the friction force is a chaotic function of the relative velocity; the chaotic behaviour may be understood in terms of a degree of freedom of motion normal to the surfaces in friction contact. Spectral analysis showed a specific structure of power spectra characteristic of quasiperiodically forced systems which has already been known from numerical experiments. A new method of predicting power spectra components is proposed.

1. INTRODUCTION

Recently we can observe a growing interest in studying forced dynamical systems with multiple incommensurate forcing terms [1–11]. Forcing at (at least) two irrationally related frequencies is common in engineering systems; indeed forcing at a single frequency is likely to be the exception rather than the rule. A fortiori, in naturally occurring dynamical systems, physical or biological, a multi-peaked spectrum of forcing is to be expected.

The dynamics of these systems is generally substantially different from that of single frequency counterparts, for example, the types of invariant sets supporting such systems are more complicated. Quasiperiodic forcing implies that the most elementary invariant sets are tori, rather than periodic orbits as in single frequency driving. In addition new invariant sets can be found, as in case of recently discovered strange nonchaotic attractors [1–11]. Strange nonchaotic attractors show some very similar properties as chaotic attractors and special methods have to be applied to distinguish them [7–9].

In a recent paper, El Naschie demonstrated that Gödel theorem implies strange nonchaotic behaviour for classical and quantum system (see ref. [23]). It is thus possible that there are more strange nonchaotic attractors than there are chaotic attractors.

The objectives of current investigations were threefold. Firstly to throw light on the rather poorly understood relationship between frictional force and relative velocity in dry sliding contact between solids. Secondly, to generate data from a real mechanical nonlinear system forced quasiperiodically and to use methods of analysis of aperiodic time series to quantify the strange behaviour of the system (chaotic or strange nonchaotic). Thirdly, to prove that the specific structure of power spectra of quasiperiodically forced systems known from numerical experiments [10] can be determined in experiments.

In Section 2 we discuss the phenomenon of dry friction and describe the experimental configuration while in Sections 3 and 4 we discuss the analysis of the experimental data. Section 3 describes properties of power spectra and introduces a new method of predicting frequency components. The analysis in Section 4 shows that dry friction force has chaotic properties. In Section 5 we interpret our results in terms of a model for the observed chaotic behaviour. Finally, our results are summarized in Section 6 where we also discuss reasons for chaotic behaviour in systems with dry friction.

2. EXPERIMENTS

The friction properties of sliding bodies are important in a large number of engineering applications. However no universal mathematical model has yet been found which satisfactorily describes these physical phenomena. The nature of the dynamic friction forces developed between objects in contact is extremely complicated and is affected by many factors such as: the frequency of the contact; the response of the interface to normal forces; the roughness of the surfaces; the wear; the lubrication; the type of the interface and others. Dynamic friction is not a simple phenomenon but comprises a set of factors different in nature and behaviour, which cannot be fully described by a simple analytical equation.

Most experimental studies [12–16] show that a dynamical friction force acts in the same direction as the relative velocity of the bodies in contact but in the opposite sense, generally according to the relationship:

$$F \sim \operatorname{sign}(v_r), \tag{1}$$

where F represents the friction force and v_r the relative velocity. The friction force has been found to be an irreversible function of the sliding velocity in all cases, in that the accelerating–decelerating branches of the friction–velocity curves are distinct in a cycle of motion. For different pairs of metals in contact, the same lubrication conditions may produce curves very distinct in shape, and even for the same combination of metals the shape, slope and separation between the two branches is very much dependent on both the dynamical properties of the test rig and the driving velocity. As a result, the experimental characteristics are not defined uniquely by the nature of contacting bodies but are functions of all the dynamical properties of the system. Experimental measurements of the dry friction force between steel and brass in contact are presented using the equipment shown in Fig. 1. The test rig provides additional facilities for dry friction tests compared with previous experiments, where in the latter the velocity of one of the contacting bodies was constant. In this test rig, all the bodies in contact are oscillating. Two simple subsystems, each with its own driving force, $F_{1,2} \cos \Omega_{1,2} t$, are coupled through the frictional contact between brass blocks placed in the bottom subsystem and held by springs in permanent contact with the steel shaft of the top subsystem. Friction forces were measured using a piezo-electric force transducer connected to a charge amplifier and a 12 bit data acquisition system in a microcomputer. Simultaneously, other system variables, for example velocities and displacements of both subsystems were measured.

In addition to the forced vertical oscillations of the masses 1 and 2 it was found that the

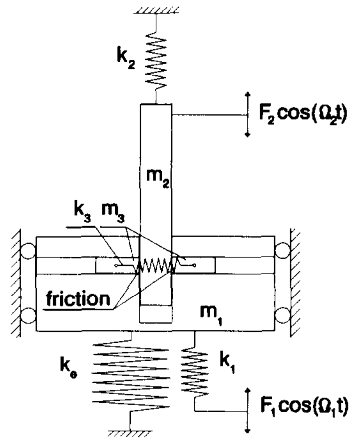

Fig. 1. The experimental rig.

masses 3 oscillate horizontally. These horizontal oscillations are caused by the roughness of the sliding surfaces of the masses 1 and 3. It is well-known that even the most polished metallic surfaces are not perfectly flat. Under magnification, one observes that these surfaces have undulations that form hills and valleys, the dimensions of which are large in comparison with molecular dimensions (Fig. 2). This horizontal degree of freedom, though very small, nevertheless appears to impose its signature on the overall dynamics of the system. A full description and discussion of the experimental technique and results can be found elsewhere [17].

3. PROPERTIES OF THE POWER SPECTRA

In our previous work [10] we found some interesting properties of frequencies spectrum of quasiperiodically forced systems which can be summarized in Fig. 3. Let us say that we have a system with forcing frequencies Ω_1 and Ω_2. Consider Ω_1 as a constant and take Ω_2 as a control parameter. When Ω_1/Ω_2 is rational, the considered system is of course periodically forced and if it is not chaotic, we observe a relatively simple frequency spectrum. Going with Ω_2 out of the point Ω_1/Ω_2 (=rational number) we observe a trifurcation of each frequency in the power spectrum; i.e. in the vicinity of each frequency component two new symmetrical components arise. Further increase or decrease of Ω_2 results in the origin of other frequency components in the described mechanism.

In this section we show that the similar properties of frequency spectra are characteristic for our experimental system and we propose a simple method for predicting frequency spectrum components.

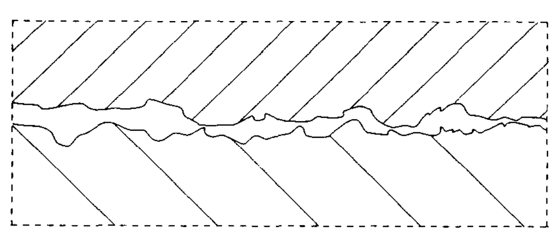

Fig. 2. A typical profile of a mild steel specimen after surface ground and polished.

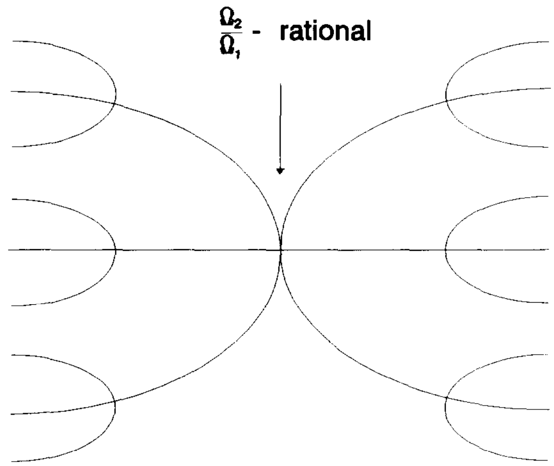

Fig. 3. A typical structure of the power spectrum of quasiperiodically forced system.

Let us concentrate on the properties of frequency spectra of displacements x_1 and x_2. Typical examples of these spectra are shown in Fig. 4(a, b). In Fig. 4(a) we have spectra obtained when the ratio Ω_1/Ω_2 is very near to the rational value of 3. In this case the power spectra have significant peaks at Ω_1, $2\Omega_1$, $3\Omega_1$ and $4\Omega_1$ and of course responses $x_1(t)$ and $x_2(t)$ are $2\pi/\Omega_1$-periodic. In Fig. 4(b) we present spectra for $\Omega_1/\Omega_2 = 3.102\ldots$ which is slightly higher than the value of the previous case. Figure 4(b) shows the existence of the other peaks which are present in the vicinity of the peaks of Fig. 4(a). New peaks appear in pairs symmetrical to the main ones, for example at frequencies: $(4\Omega_1 - \Omega_2)$ and $(\Omega_2 - \Omega_1)$, and, $(7\Omega_1 - 2\Omega_2)$ and $(2\Omega_2 - 5\Omega_1)$—both marked in Fig. 4(b). Generally, we observe a structure of power spectra similar to the one described in Fig. 3 which is characteristic for the whole range of system parameters [17].

The described structure of power spectra allows us to develop the following method of predicting components of power spectra. This method is based on the following construction. We plot the diagram of the possible frequency spectra in the function of the excitation frequency—Ω_2 with other frequency Ω_1 fixed—Fig. 5. The diagram represents values of the frequencies of the type $f_{ij} = \pm i\Omega_1 \pm j\Omega_2$, which are simply the lines. The level of complication of the picture depends on the maximum values of the numbers i and j selected. The structure is repeatable, this means that decreasing the window size similar patterns can be obtained as shown in Fig. 5(a) and Fig. 5(b). The number of i and j which is sufficient for the analysis of our experimental results is about 21.

Of course, not all of the predicted peaks are always present in power spectra but the accuracy of the prediction is high as it is shown in Fig. 6(a) and Fig. 6(b) where we have compared the values of the predicted peaks with those found in experiments.

4. BEHAVIOUR OF THE FRICTION FORCE

Typical examples of friction force vs relative velocity dependence in a quasiperiodically excited two degrees of freedom system are reproduced in Fig. 7(a–c). In Fig. 7(a) the dependence is approximately described by equation (1). This case was obtained for the values of Ω_1 and Ω_2 close together ($\Omega_1 = 10.051$ Hz, $\Omega_2 = 14.61$ Hz). In Fig. 7(b) and 7(c) the cases when the value Ω_2 is substantially larger than Ω_2 [$\Omega_1 = 10.51$ Hz, $\Omega_2 = 30.83$ Hz—Fig. 7(b) and 35.03 Hz—Fig. 7(c)], the friction–velocity relation seems to

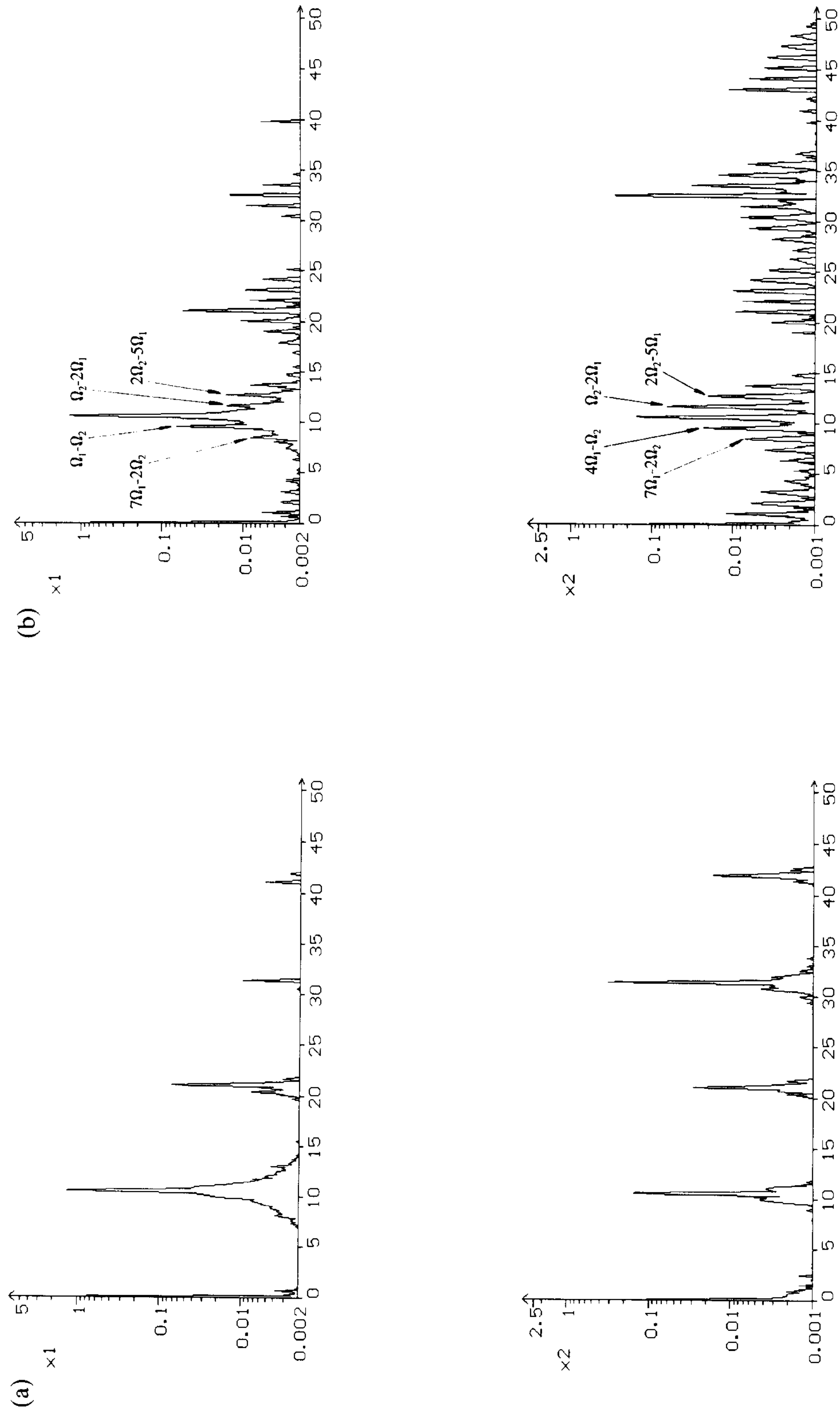

Fig. 4. Examples of the power spectra of $x_1(t)$ and $x_2(t)$: (subsystem 1—the top part, subsystem 2—the bottom part). (a) $\Omega_1 = 10.45$ Hz, $\Omega_2 = 31.45$ Hz; (b) $\Omega_1 = 10.52$ Hz, $\Omega_2 = 32.62$ Hz.

(a)

(b)

(c)

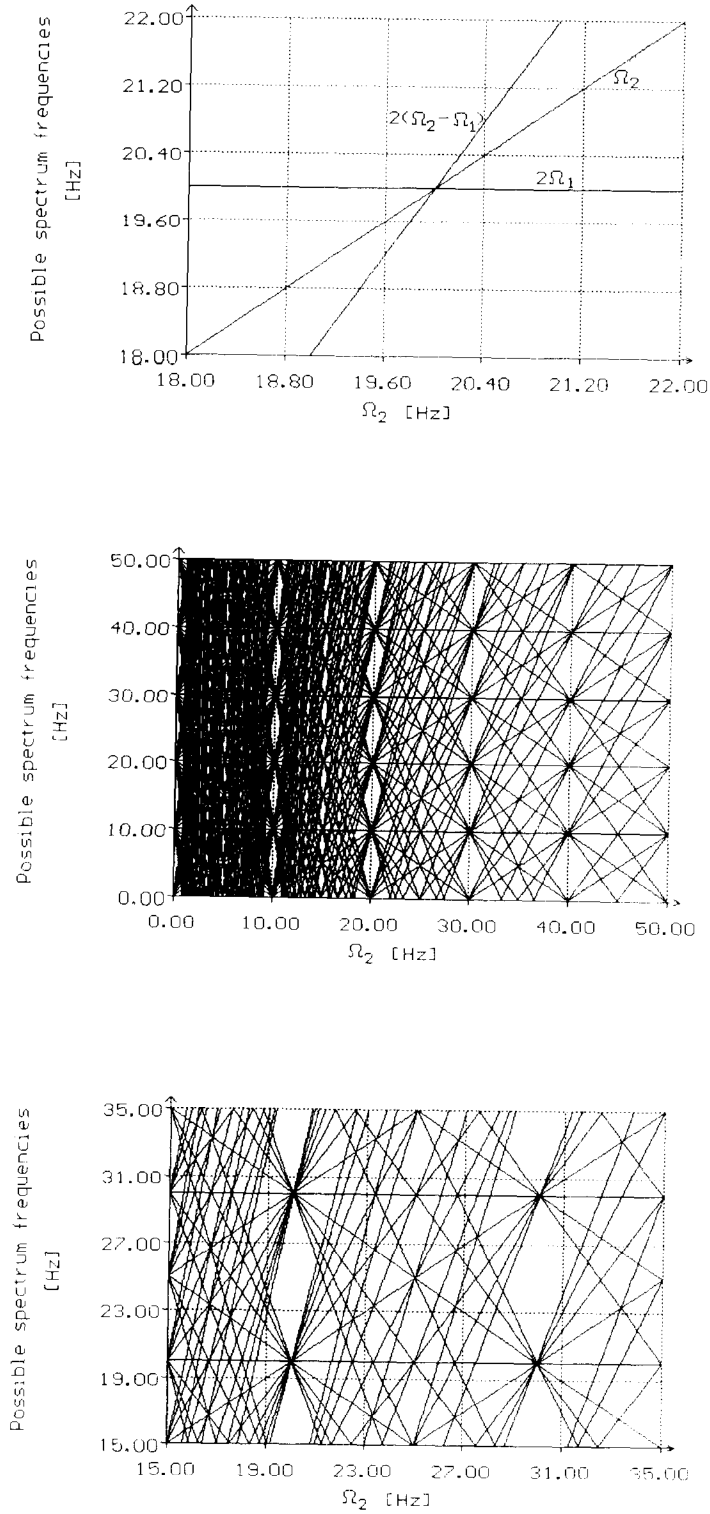

Fig. 5. Construction of the prediction method ($\Omega_1 = 10.0$ Hz): (a) main idea; (b) plot for $i = j = 11$; (c) enlarged part of the plot for $i = j = 11$.

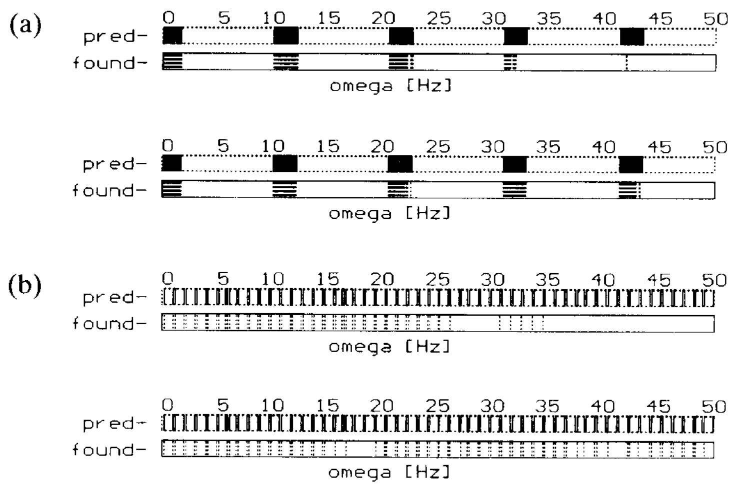

Fig. 6. Predicted and observed components of the power spectra: (subsystem 1—the top part, subsystem 2—the bottom part). (a) $\Omega_1 = 10.45$ Hz, $\Omega_2 = 31.45$ Hz, (b) $\Omega_1 = 10.52$ Hz, $\Omega_2 = 32.62$ Hz.

be completely different from that of equation (1); force–velocity changes are now highly unpredictable and complicated.

The behaviour displayed in Fig. 7(b, c) is clearly aperiodic and a first objective of analysis is to determine the character of the attractor and behaviour of trajectories near it, i.e. we try to argue if the relation between frictional force and relative velocity is chaotic (exponentially sensitive to initial conditions) or only aperiodic.

Our approach follows that of Grassberger and Procaccia [18]. We constructed an m-component 'state' vector $\mathbf{X}_i$ from a time series of friction force $F(t)$ as

$$\mathbf{X} = \{F_1(t_1), F_2(t_1 + \tau), \ldots, F_m(t_1 + (m - 1)\tau)\},$$

where τ is an appropriate time delay (of the order of characteristic physical time scales) and used the correlation integral defined for N vectors distributed in an m-dimensional space as a function of distance r:

$$C(r, m) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^{N}\sum_{j=1}^{N} \Theta(r - |X_i - X_j|).$$

where Θ is the Heaviside step function. If the number of points is large enough, as assumed above, this distribution will obey a power-law scaling with r for small r: $C(r, m) \sim r^{\nu}$, where ν is the correlation dimension. As we increase m, the correlation dimension is seen to converge to its true value.

The results of this procedure applied to the data of Fig. 7(b) are shown in Fig. 8. We find that the correlation dimension converges to a value greater than 6 for embedding dimension $m = 8$, and appears not to change for further increase of m. The data of Fig. 7(c) give results which are very similar, the value of n at $m = 8$ differs by only 0.1.

5. A MODEL FOR THE OBSERVED BEHAVIOUR

The equations for the vertical motion of the two oscillators take the form:

$$\begin{aligned} (m_1 + m_3)\ddot{x}_1 + f(x_i, \dot{x}_i, N) + k_1 x_1 &= F_1 \cos(\Omega_1 t), \\ m_2\ddot{x}_2 + f(x_i, \dot{x}_i, N) + k_2 x_2 &= F_2 \cos(\Omega_2 t), \end{aligned} \tag{2}$$

where N is the total normal force acting at the sliding contacts.

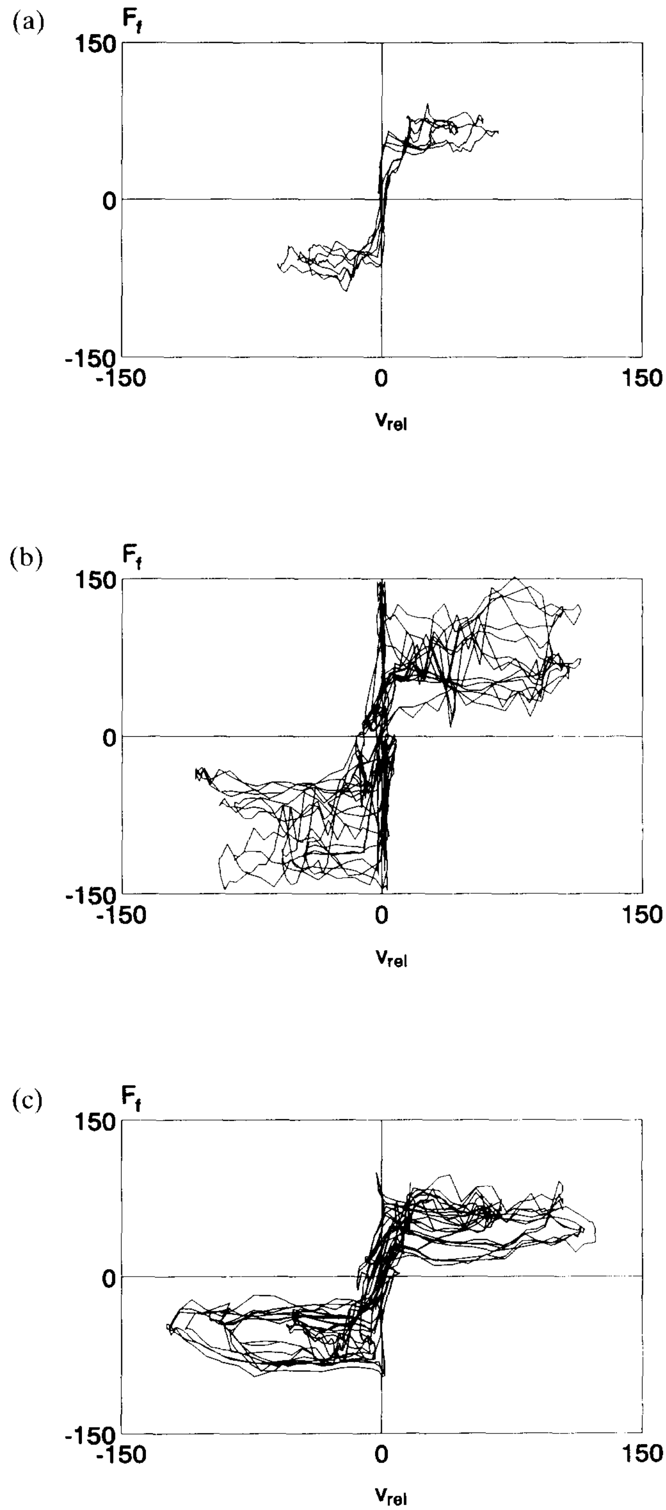

Fig. 7. Examples of the friction force–relative velocity diagrams: (a) $\Omega_1 = 10.51$ Hz, $\Omega_2 = 14.01$ Hz; (b) $\Omega_1 = 10.51$ Hz, $\Omega_2 = 30.83$ Hz; (c) $\Omega_1 = 10.51$ Hz, $\Omega_2 = 35.03$ Hz.

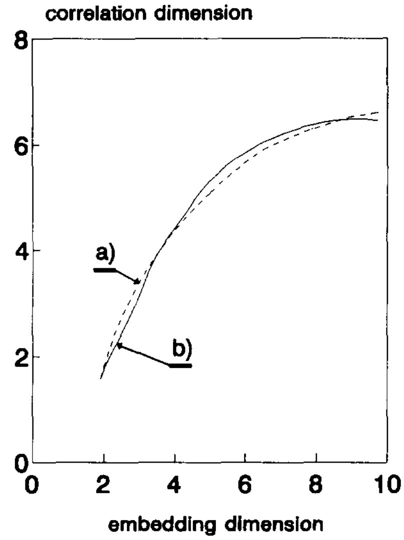

Fig. 8. Correlation dimension vs embedding dimension plots: (a) friction force time series of Fig. 7(b); (b) friction force time series of Fig. 7(c).

If we assume N to be constant, flow defined by this equation occupies a six-dimensional phase space and can be described by six one-dimensional Lyapunov exponents. Two of them, connected with the forcing, are explicitly zero. Of the other four at least two must be negative to reflect the dissipative nature of the system. Hence the Lyapunov dimension, given by:

$$d_{\mathrm{L}} = j + \sum_{i=1}^{j} \frac{\lambda_i}{|\lambda_{j+1}|}, \tag{3}$$

where j is the largest number of Lyapunov exponents for which the sum $\sum_{i=1}^{j}\lambda_i$ is non-negative, is at most 4 if there are no positive Lyapunov exponents. A value of d_{L} greater than 4 indicates that at least one Lyapunov exponent must be positive and that the system is chaotic.

Kaplan and Yorke [19] have suggested the equality of the Lyapunov dimension and the information dimension, d_{I}, and it has been shown [18] that the correlation dimension ν and information dimension d_{I} satisfy:

$$\nu < d_{\mathrm{I}}. \tag{4}$$

Hence the experimental data suggest that, since $\nu > 6$ and hence $d_{\mathrm{I}} > 6$ the model represented by equations (2) is inadequate to describe the dynamics, since its phase space dimension is too low to produce such a value for d_{I}.

We propose that the reason for the inadequacy is the assumption in equations (2) of constant N. A more adequate model must allow for variations in N associated with small irregularities in the surfaces in contact and motion normal to the plane of contact. We propose that N can vary and is connected to the system (2) through an equation of the form:

$$m\ddot{x}_3 + kx_3 - N = \text{constant}, \tag{5}$$

governing the dynamics of this normal motion, where $x_3(x_1, x_2)$ is the local departure of the separation of the brass slides from its mean value as a result of surface roughness.

This equation, added to equations (3) produces an eight-dimensional system. An embedding dimension of 8 appears to give a good approximation for the value of ν and suggests that consideration of the degree normal to the direction of slipping is important in modelling the friction force. Calculations of ν for other values of Ω_1 and Ω_2 support the converged value close to 6.70 for an embedding dimension of 8.

Finally, equations (2) and (5) have eight one-dimensional Lyapunov exponents. The greatest possible Lyapunov dimension for the case where there is no positive Lyapunov exponent is 6. Hence our experimental data suggest that the behaviour of the system is chaotic.

6. CONCLUSIONS

Our research showed that the nonlinear phenomena investigated on the basis of the low-dimensional system described by the differential equations are present in real mechanical system.

Particularly, we found that a specific structure of the power spectra of quasiperiodically forced systems which was previously determined in numerical experiments [8] is present in real experimental systems as well, and seems to be general. This structure allows us to develop a graphical method of prediction of main frequencies in the power spectrum. It can also be used to identify routes to chaos. Determination of the significant peak in the power spectrum which is not predicted by our method can be taken as an indicator of Hopf bifurcation. We have to note here that Hopf bifurcation in two frequency forced systems creates the third incommensurate frequency and three frequency torus as an invariant set. It is well-known that such a torus is very unstable [1] and its break creates a strange attractor. The similar method which considers frequencies $(i\Omega_1 + j\Omega_2)/2^n$; $i, j, n = 1, 2, \ldots$ can be developed to identify torus doubling route to chaos.

The friction force properties were found to be not simple, as most of the previous assumptions and very complicated dependencies on relative velocity are present. The results suggest that for some system parameters friction force is chaotic. As it is impossible to describe the friction–velocity dependence by a deterministic relationship a stochastic model should be more suitable. The simplest stochastic model which can describe chaotic behaviour of friction force can be the one presented in Fig. 9(a–c). For positive relative velocities the friction force is a random variable in area A, while for negative relative velocity, the friction force is a random variable from area B. Depending on system parameters, areas A and B can evolve from these in Fig. 9(a) where the proposed model can be considered as a noisy version of relation (1), through these in Fig. 9(b) to these in Fig. 9(c) where only the law that the friction force has the same sign as a relative velocity is fulfilled.

Our investigations showed that in the systems with dry frictions there are at least two sources of chaos:

(1) nonlinearity in the system (even as simple as one described by equation (1);
(2) chaotic properties of dry friction.

The first source was investigated in a number of papers [16, 20–22] both experimentally and numerically. The second source depends on the materials and configuration properties of the system and to the best of our knowledge is mentioned here for the first time. It should be noted here that this source is hard to eliminate in practical systems and requires further investigations.

The inclusion of a degree of freedom normal to the plane of slipping is important in

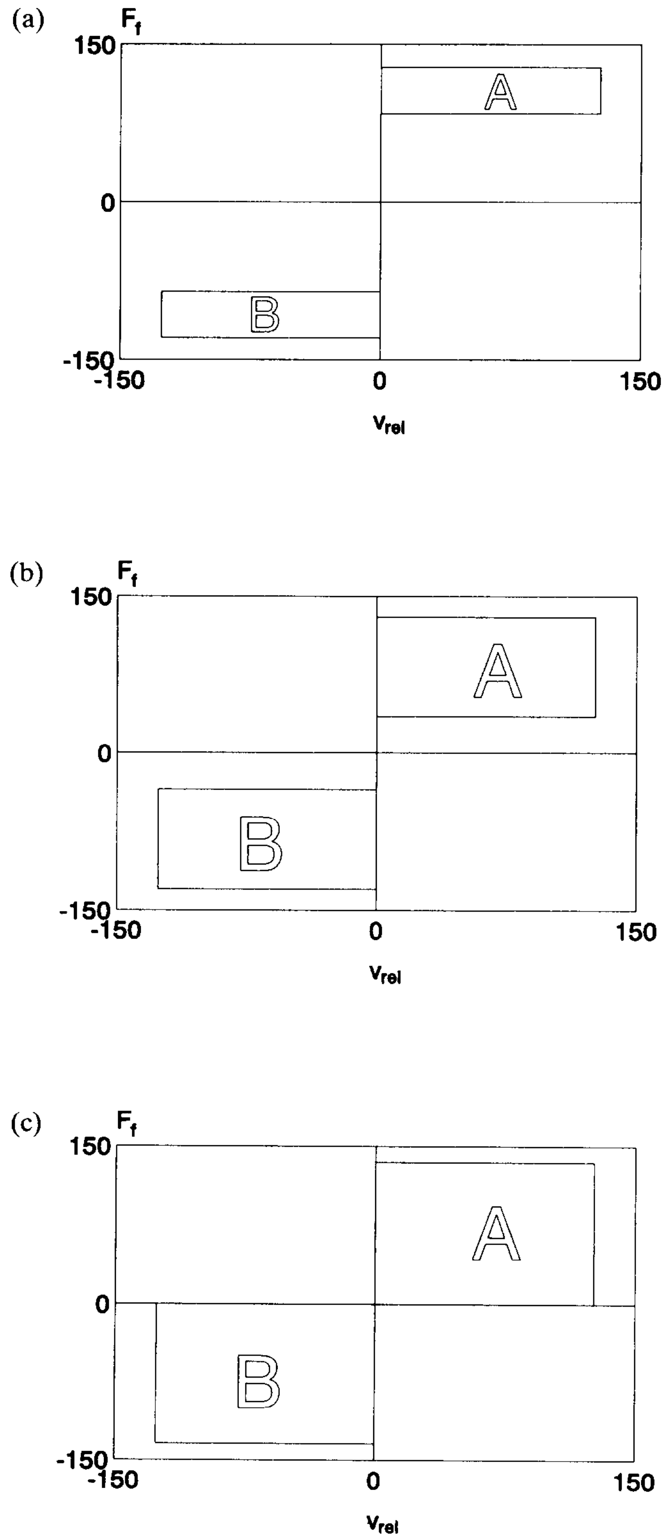

Fig. 9. Stochastic models of the dry friction force.

modelling the experimental consequences of the dry friction. This so-called normal degree of freedom was suggested as important factor in identification of the dry friction long ago [13, 14]. Our investigations give evidence of its importance based on the theory of nonlinear dynamics.

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