Generating strange nonchaotic trajectories

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We show a simple procedure which allows us to generate a strange nonchaotic trajectory. It can explain properties of strange nonchaotic attractors.

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Recently there has been much interest in the dynamical properties of quasiperiodic forced systems. Forcing at (at least) two irrationally related frequencies is common in naturally occurring dynamical systems, physical or biological, where a multipeaked spectrum of forcing is to be expected. The dynamics of these systems can differ substantially from that of their single-frequency-driven counterparts. In addition to the well-known dynamical behaviors that result, two-frequency, three-frequency quasiperiodic, and chaotic attractors it is possible to observe another type of behavior leading to strange nonchaotic attractors [1–10]. Strange nonchaotic attractors (SNA's) are characterized by fractal structure but typical nearby trajectories on it do not diverge exponentially with time. It has been shown that SNA's are typical for quasiperiodically forced systems. They have been found not only in a number of numerical experiments [1–7], but in experimental systems as well [8–10]. Until now there have been no simple models that can explain the behavior of trajectories on strange nonchaotic attractors.

Generally, SNA's can occur in the four-dimensional phase space of dissipative systems. In this Brief Report we show a controlling technique that allows us to generate a strange nonchaotic trajectory by making small changes in the parameters of the three-dimensional system. Our method is applicable to the systems in which behavior depends on a control parameter c such that they have a chaotic attractor for one value of c, say, c_1, and a strange repeller together with a periodic attractor for the other value of c, c_2. Systems with a strange repeller exhibit transient chaos [11,12]. Trajectories started from randomly chosen initial points then approach the attractor with probability 1. Before reaching it, however, they might come close to the strange repeller and stay in its vicinity for shorter or longer periods of time. Long-lived chaotic transients are often present around crisis configurations [11], at parameter values just beyond the disappearance of the chaotic attractor. It is worth mentioning that systems with fractal basin boundaries [13] are also accompanied by transient chaos since such boundaries are, in general, the stable manifolds of a chaotic repeller. Computation of transient Lyapunov exponents for the systems with chaotic repellers often shows that their values become nonpositive long before the transient dies, i.e., before the trajectory reaches the attractor [14,15]. (Transient Lyapunov exponent is not a common term, as the classical definition as a limit for $t \to \infty$ cannot be generalized to define a time-dependent quantity. By transient Lyapunov exponent, we mean a value obtained for a finite t not large enough to ensure a satisfactory reduction of the fluctuations but small enough to reveal slow trends. The same definition can be found in [14].)

The main idea of our method is described in Fig. 1. Let us consider two trajectories that start from nearby initial conditions A' and A'', which lie on or close to a strange chaotic attractor. For $t_0 < t < t_1$ these trajectories represent the evolution of the system for a value of control parameter $c = c_1$, and for $t_1 < t < t_3$ the evolution is shown for control parameter $c = c_2$. In the first time interval we observe exponential divergence of trajectories described by the positive Lyapunov exponent $\lambda(t) > 0$. At time $t = t_1$ we are changing a value of the control parameter from c_1 to c_2. For a new value of the control parameter our system has a strange repeller and we observe first a further divergence of trajectories and a positive transient Lyapunov exponent $\lambda(t) > 0$ for $t_1 < t < t_2$. At time t_2 a transient Lyapunov exponent changes sign from positive to negative and for $t_2 < t < t_3$ we observe a convergence of trajectories. At time t_3 we are changing a value of the control parameter to c_1 again, etc. If t_3 is chosen such that the period of time $t_3 - t_2$ is not sufficient for a system to reach periodic attractor and

$$\int_{t_0}^{t_2} \lambda(t)dt \approx \int_{t_2}^{t_3} \lambda(t)dt \tag{1}$$

then we do not observe divergence of trajectories in time. As a part of the trajectory evolves on the strange chaotic attractor the switches between c_1 and c_2 will take place in different points of phase space so the trajectory is

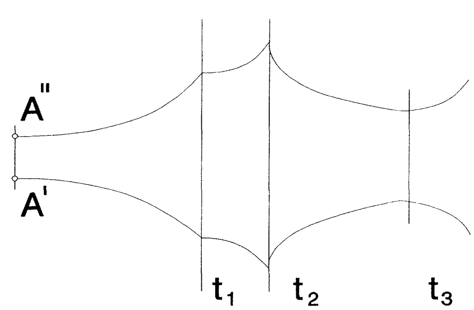

FIG. 1. Behavior of nearby trajectories.

aperiodic (not t_3 periodic) and has properties typical of trajectories on strange nonchaotic attractors.

As an example of our method let us consider

$$\ddot{x} + a\dot{x} - [1 + c\cos(\omega t)]x + bx^3 = 0 , \tag{2}$$

where a, b, c, and ω are constant. The examples of this equation are found in many applications of mechanics, particularly in problems of dynamical stability of elastic systems [16,17]. In this Brief Report we took $a = 0.1$, $b = 1$, $\omega = 1$, and c as a control parameter.

Equation (2) has three Lyapunov exponents: one of them is always zero, one is always negative, and the third one can change its sign with the change of the control parameter c. The value of this Lyapunov exponent is responsible for exponential divergence or convergence of nearby trajectories and in the rest of this Brief Report we shall investigate only the value of this Lyapunov exponent λ. If λ is negative we have the limit-cycle attractor, if λ is positive we have a strange chaotic attractor, and if $\lambda = 0$ a torus is an attractor.

The behavior of Eq. (2) has been investigated numerically in [15], where Eq. (2) has been integrated by the fourth-order Runge-Kutta method with the integration step being $T_1/200$, where $T_1 = 2\pi/\omega$ and Lyapunov exponents have been obtained using the method of Wolf *et al* [18]. The plot of λ vs control parameter c is shown in Fig. 2 as c changes from 0 to 0.5. If c is small, λ is negative, so Eq. (2) does not show sensitive dependence on the initial conditions. When c is increasing up to about 0.348, λ changes suddenly from negative to positive values and the behavior of Eq. (2) is chaotic. In the interval $c \in (0.256, 0.348)$ we observe long transient aperiodic trajectories without sensitive dependence on the initial condition, i.e., the transient Lyapunov exponent $\lambda(t)$ decreases and turns negative far before transients have died. The lifetime of the observed transient is relatively long $[(10^6 - 10^{10})T_1]$ in comparison to the lifetime of transient chaos (when the transient Lyapunov exponent is positive) which is much shorter (about $10^3 T_1$).

In our control procedure we took one value of c for which the behavior of the system is chaotic and one value of c for which Eq. (2) has the above-mentioned transient behavior. In numerical investigations $c_1 = 0.35$ and $c_2 = 0.34$, $t_1 = 10^5 T_1$ and $t_3 = 3 \times 10^5 T_1$ have been taken and we consider the behavior of a system:

$$\ddot{x} + a\dot{x}[1 + c(t)\cos(\omega t)]x + bx^3 = 0 , \tag{3}$$

where $c(t) = c_1$ for $t_0 < t < t_1$ and $c(t) = c_3$ for $t_1 < t < t_3$. Generally, T_1 and $T_2 = t_3$ are incommensurate so Eq. (3) has four-dimensional phase space $(x, \dot{x}, \omega t, (2\pi/t_3)t)$ and can have a strange nonchaotic attractor. In Fig. 3 we show the plot of transient Lyapunov exponent $\lambda(t)$ vs t. Since the function $c(t)$ is discontinuous, in numerical calculations we took its Fourier-series approximation with 100 components. In this figure the regions of divergence and convergence of nearby trajectories are visible. As the value of the Lyapunov exponent averaged over time T_2 is negative and close to -0.001 we do not observe exponential divergence of trajectories in time. This result shows that applying our control procedure we manage to build an aperiodic trajectory which is predictable in the sense that nearby trajectories do not diverge exponentially as it is shown in Fig. 4. Figure 4 presents the distance $\Delta - |x'(t + nT_2) - x''(t + nT_2)|$; where $n = 1, 2, \ldots, x'$ is

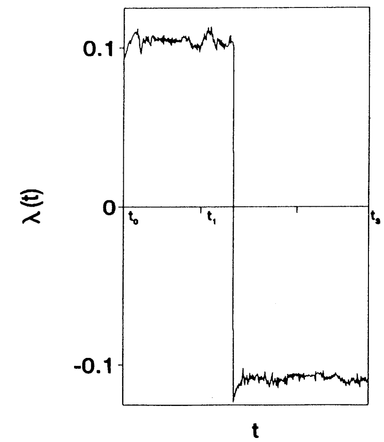

FIG. 3. Transient Lyapunov exponent $\lambda(t)$ for Eq. (3) vs time.

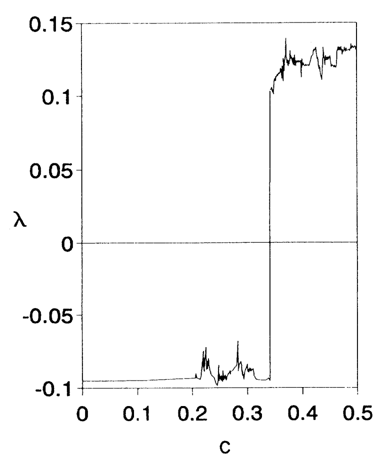

FIG. 2. Lyapunov exponent λ for Eq. (2) vs control parameter c.

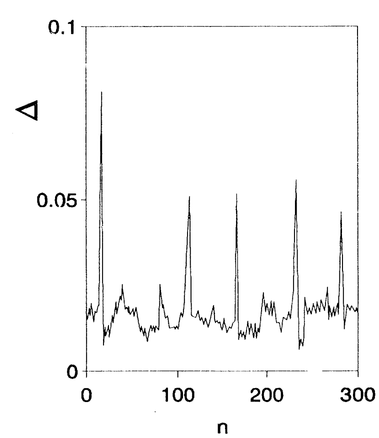

FIG. 4. Distance between nearby trajectories of Eq. (3) vs time.

a trajectory for initial conditions $x(0)=.01$, $\dot{x}(0)=0$, and x'' is a trajectory for $x(0)=0.101$, $\dot{x}(0)=0$.

Our procedure allows us to build the simplest model of a strange nonchaotic attractor and explain the behavior of trajectories on it. Periods of time when we observe divergence of nearby trajectories (positive maximum Lyapunov exponent) and periods of time when trajectories converge (negative Lyapunov exponents) as in our construction seem to be necessary for all trajectories on strange nonchaotic attractors. However, in most of the SNA's these periods do not have to be so well defined as in our model.

The method presented here can be applied only to the systems which display a specific *a priori* known behavior. This requires the knowledge of equations of motion and practically eliminates our method as a method of controlling chaos in the sense of the Ott-Grebogi-Yorke method [19–26]. However, as an effect of applying this method it is possible to obtain an aperiodic trajectory which is different from the original chaotic trajectory in that it can be predictable. The method of generating a strange nonchaotic trajectory described here can be applied for designing an aperiodic orbit which can be used as an input to the systems where an aperiodic but predictable trajectory is advantageous. In mechanical systems, gear boxes are the classical examples of such a system (aperiodic forcing reduces fatigue of materials).

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