Mechanism of noise-induced resonance

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Based on approximate methods of nonlinear oscillations, we give a possible explanation of the phenomenon of noise-induced resonance.

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Stochastic resonance is a phenomenon which is characteristic for systems forced by a sum of noise or chaotic signal and a weak periodic signal [1]. Under appropriate conditions a weak periodic signal can be amplified by noise or chaotic signal.

Recently Bartussek, Hanggi, and Jung studied the response of a noise-driven absorptive optical bistable system which is subjected to deterministic periodic perturbations of the incident light intensity [2]. Beside classical stochastic resonance they found a phase-sensitive resonance phenomenon which virtually eliminates or significantly decreases the higher harmonics. They called this phenomena noise-induced resonance.

In this paper we proposed the possible explanation of noise-induced resonance. Our mechanism based on approximate methods of nonlinear oscillations is similar to the one used in [3] for dynamical description of stochastic resonance. We consider a particular yet representative case of Duffing's equation driven by a weak periodic force plus a band-limited white noise.

$$\ddot{x}+a\dot{x}+bx^3=f(t)+A\cos\Omega t \tag{1}$$

where a, b, A, and Ω are constants. $f(t)$ is a zero-mean stochastic process with spectral density

$$\begin{aligned} &s(\nu)=\delta/(\nu_{\max}-\nu_{\min}), \quad \nu\in(\nu_{\min},\nu_{\max}) \\ &0, \quad \nu\notin(\nu_{\min},\nu_{\max}) \end{aligned} \tag{2}$$

where δ is a noise intensity and $[\nu_{\min}\ \nu_{\max}]$ is an interval of considered frequencies. Equation (1) is a single-well system which can be considered for example as a mathematical model of a nonautonomous second order circuit [4].

First consider Eq. (1) without random signal $[f(t)=0]$. In this case one can find an approximate periodic solution in the form

$$\begin{aligned} x(t)=&C_1\cos(\Omega t+\phi_1)+C_2\cos(2\Omega t+\phi_2) \\ &+C_3\cos(3\Omega t+\phi_3) \end{aligned} \tag{3}$$

where the constants C_{1-3} (amplitudes of the first three harmonics) and ϕ_{1-3} (phase difference between oscillations and periodic forcing) can be estimated by intersecting Eq. (3) into Eq. (1) and applying, for example, the harmonic balance method [4,5]. Although the application of the described method requires numerical calculations in the theory on nonlinear oscillation it is called an analytical one as no direct numerical integration is necessary. Nowadays algebraic calculations in this method can be easily performed using computer algebra systems. We used Mathematica Typical relations between C_{1-3}, ϕ_{1-3} and A can be found in [4,5]. Here as we are interested in the higher harmonics we present a typical stable curve for C_3 which is shown in Fig. 1 (curve *a*). It it visible that although generally C_3 increases with the increase of A, it is the region of A $(A_1<A<A_2)$ in which C_3 rapidly decreases.

Now consider a complete form of Eq. (1) (with periodic and random forcing). It is well known that random forcing of the form of band-limited white noise (its realizations) can be approximated by a sum of N harmonics,

$$f(t)=\delta\sum_{i=1}^{N}\cos(\nu_i+\Psi_i) \tag{4}$$

where ν_i and Ψ_i are independent random variables [5]. Frequencies ν_i are described by

$$\nu_i=(i-1/2)\Delta\nu+\delta\nu_i+\nu_{\min} \tag{5}$$

where $\Delta\nu=(\nu_{\max}-\nu_{\min})/N$ and Ψ_i are independent random variables with uniform distribution on the interval $[0,2\pi]$. For a given realization of random forcing $f(t)$ the above approximation allowed us to consider Eq. (1) as a deterministic system. Following an analytical approach introduced in [5] one can assume the solution of Eq. (1) in

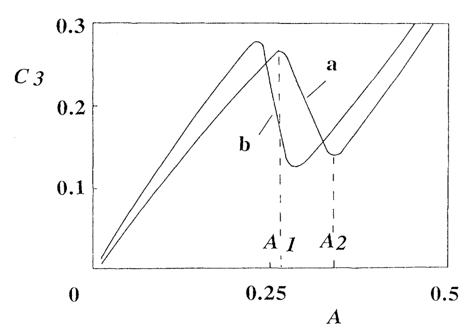

FIG. 1. $C_3(A)$ curves for Eq. (1): $a=0.04$, $b=10.0$, Ω 10.0; curve *a*, noiseless system $f(t)=0$; curve *b*, system with noise $\delta=0.1$, $\nu_{\min}=0$, $\nu_{\max}=50$.

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the form

$$x(t)=C_1\cos(\Omega t+\phi_0)+C_2\cos(2\Omega t+\phi_2)+C_3\cos(3\Omega t+\phi_3)+\sum_{i=1}^{N} C_i\cos(\nu_i t+\Psi_i) \quad (6)$$

where C_i and ϕ_i $(i=0,1,\ldots,N)$ are constant. Inserting Eq. (6) into Eq. (1) it is possible to compute all constants in Eq. (6). Typical C_3 curve for $a=0.04$, $b=10.0$, $\Omega=10.0$, $N=50\,000$, and $\nu_{\min}=0$, $\nu_{\max}=50$ is shown in Fig. 1 (curve b). Comparing curve a with curve b in Fig. 1 one finds that an addition of noise slightly changes the $C_3(A)$ plot but for some values of A (for example $A=0.28$) we can observe a significant decrease of C_3. In Fig. 2 we present the variation of amplitude C_3 for different noise intensity δ and constant amplitude of periodic signal $A=0.28$ where the effect of noise-induced resonance is clearly visible. In this plot we showed also the level of noise in the solution $x(t)$ of Eq. (1) estimated as a maximum of the last component in Eq. (6).

The above observation allows us to state that the noise-induced resonance in Duffing's equation is caused by the noise-induced shift of 3Ω-resonance curve $C_3(A)$, particularly by the shift of the region in which C_3 decreases. This shift allows C_3 to decrease with the increase of noise intensity δ as shown in Fig. 2. In some cases C_3 can be smaller than noise level in solution $x(t)$. This explains why signal amplification

$$\eta_3=4\frac{C_3^2}{A^2} \quad (7)$$

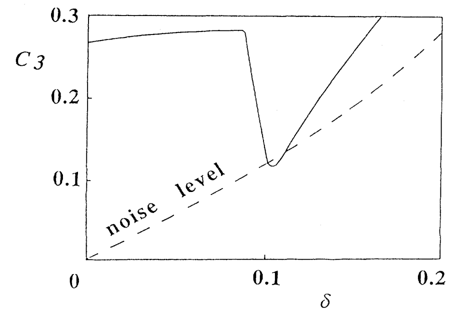

FIG. 2. $C_3(\delta)$ curve for Eq. (1): $a=0.04$, $\Omega=10.0$, and $A=0.28$.

consider in [3] can be zero at noise-induced resonance.

In conclusion, the application of approximate analytical methods of nonlinear oscillations allows one to explain a mechanism of noise-induced resonance. We found evidence that noise-induced resonance in Duffing's equation can be explained by the noise induced shift of resonance curve.

It should be added here that the similar mechanism of noise-induced resonance can be applied to the systems forced by chaotic and periodic force instead of considered here noise and periodic external forces. this result will be published later.

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