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On Transverse Stability of Synchronized Chaotic Attractors

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Abstract – We consider the stability of the synchronized chaotic attractor of two identical, symmetrically coupled chaotic systems. The transverse stability of the synchronized chaotic attractor can be investigated based on the linearization of the transverse flow. In a numerical study we also demonstrate the applicability of an approximate approach to understand monotonic and asymptotic stability (synchronization) as well as on-off (chaos-hyperchaos) intermittency. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

There has been a considerable study of synchronized chaotic systems recently [1-6]. In most synchronization procedures two *n*-dimensional chaotic systems $\dot{x} = f(x)$ and $\dot{y} = f(y)$ are coupled in some way. This coupling creates a 2*n*-dimensional augmented dynamical system $\dot{X} = F(X)$. Full phase space of this augmented system possesses a smooth invariant *n*-dimensional manifold *M*. Any orbit originating in this manifold stays there forever, approaching the chaotic attractor *A* of either of the dynamical systems, say $\dot{x} = f(x)$, as $t \to +\infty$. It has been shown that two chaotic systems can synchronize only when the chaotic attractor of the *n*-dimensional dynamical system is stable in the 2*n*-dimensional phase space of the augmented system $\dot{X} = F(X)$ [7-16]. This type of stability is referred to as transverse stability. The spectrum of Lyapunov exponents of the augmented system can be divided into two subsets: $\lambda^{(1)}$ representing motion which is tangential, and $\lambda^{(2)}$ transverse, to the invariant manifold. It has also been shown that if all transverse Lyapunov exponents are negative, then the chaotic attractor *A* is stable [7-16].

In this paper we investigate transverse stability of the chaotic attractor A based on the properties of the flow transverse to it and its linearization. We consider a system consisting of two coupled identical subsystems governed by

$$\dot{x} = f(x) + D(y - x)$$

 $\dot{y} = f(y) + D(x - y),$
(1)

where $x, y \in \mathbb{R}^{2n}$, $D \in \mathbb{R}$, D > 0, and we assume that the decoupled subsystems $\dot{x} = f(x)$ and $\dot{y} = f(y)$ have an asymptotically stable chaotic attractor A in an invariant manifold M defined by the synchronized state x = y.

The outline of this paper is as follows. Section 2 describes the basic bifurcations which are characteristic for eqn (1). In Section 3 we introduce the concept of transverse linearization in the neighbourhood of the chaotic attractor. Section 4 presents a numerical

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study of an example where two Rossler systems were coupled together. Finally, we summarize our results in Section 5.

2. TRANSVERSE BEHAVIOUR IN THE NEIGHBOURHOOD OF THE CHAOTIC ATTRACTOR

For the analysis of the transverse stability of a chaotic attractor A we introduce the new variables

$$R(t) = \frac{1}{2}(x(t) + y(t)), \quad e(t) = \frac{1}{2}(x(t) - y(t)).$$
(2)

With this transformation one replaces system (1) with the equivalent system

$$\dot{R} = \frac{1}{2}(f(R+e) + f(R-e)) = h(R,e)$$
 (3a)

$$\dot{e} = \frac{1}{2}(f(R+e) - f(R-e)) - 2De = g(R, e; D).$$
 (3b)

The variable e(t) describes the evolution transverse to the *n*-dimensional invariant manifold M, explained in Section 1, while the variable R(t) yields the evolution of the theoretical 'centre-of-mass' position of the combined subsystems in the presence of transverse motion. In the limit of no transverse motion, R(t) describes the evolution in the *n*-dimensional invariant subspace M.

The spectrum of Lyapunov exponents of eqn (3) are now divided into two subsets: the set $\lambda^{(1)}$ is associated with the evolution of R(t) describing the dynamics on (or close to) the manifold M, while the other set $\lambda^{(2)}$ describes the propagation of the perturbations orthogonal to M.

In terms of e(t) we can define synchronization in the following way. Two chaotic systems $\dot{x} = f(x)$ and $\dot{y} = f(y)$ are synchronized if

$$\lim_{t \to +\infty} e(t) = 0. \tag{4}$$

If the limit (4) is fulfilled for all initial values in the neighbourhood of e(t) = 0, then the attractor A is asymptotically stable.

If additionally the following inequality holds

$$\frac{\mathrm{d}|\boldsymbol{e}(t)|}{\mathrm{d}t} < 0,\tag{5}$$

then the synchronization is monotonic, i.e. after each perturbation the distance between the actual trajectory and the attractor is a decreasing function of time. In the case of monotonic synchronization for all initial values in the neighbourhood of e(t) = 0 we refer to the chaotic attractor A as monotonically stable. Of course, the monotonic stability is a special case of asymptotic stability.

Let us next describe the various stability transitions of the chaotic attractor A in terms of the control parameter D. For sufficiently large D, say $D > D_3$, the chaotic attractor A is monotonically stable, the strongest type of stability discussed here. Synchronization is monotonically achieved for all initial conditions in the neighbourhood of A. Estimates of D_3 should be useful information for safe synchronization.

Most analyses, however, have in the present context been dealing with asymptotic stability. It can be shown [1] that the chaotic attractor A is asymptotically stable (synchronization is achieved for all initial conditions in the neighbourhood of A) for $D > D_2 = \lambda_1^{(1)}/2$, where $\lambda_1^{(1)}$ is the leading Lyapunov exponent of the attractor. At $D_2 = \lambda_1^{(1)}/2$ we have a bifurcation where the chaotic attractor A looses its asymptotic stability.



CHAOS



Fig. 1. Schematic illustration of the order in which stability bifurcations of the synchronized, chaotic attractor occur on an arbitrary scale of the synchronization control D.

For $D < D_2$ the attractor is still stable (there exists a 2*n*-dimensional basin b(A) with a positive Lebesque measure). However, it is not asymptotically stable as b(A) does not contain all neighbourhoods of A. In this case, for a typical trajectory of (3), all transverse Lyapunov exponents $\lambda^{(2)}$ are negative. However, there are still initial conditions dense in the attractor for which one of the transverse exponents is positive. In the region $D < D_2$, the chaotic attractor A can also have a locally riddled basin [17-20], if there is an $\varepsilon > 0$ such that for every point $x \in b(A)$ any arbitrarily small ball centred on x contains a set of positive measure whose orbits exceed a prescribed distance ε from A. In our case there would be riddled basins of two attractors: the chaotic attractor A in case of synchronization, and the hyperchaotic attractor in case of no synchronization.

With a further decrease of the control parameter D our system undergoes a blowout bifurcation (chaos-hyperchaos transition) [7-16] at, say $D = D_1$. After this bifurcation, one of the transverse Lyapunov exponents for a typical orbit on the attractor will always be positive. The chaotic attractor A becomes a chaotic saddle. A chaotic invariant set Ahaving a dense orbit is a chaotic saddle [21] if there is a neighbourhood U of A such that $b(A) \cap U$ is greater than A but has zero Lebesque measure. Here $(D < D_1)$ we observe the phenomenon of on-off intermittency (chaos-hyperchaos intermittency) [7-16] in which a typical phase space trajectory spends some of the time in the neighbourhood of the attractor A and occasionally burst away from it. For on-off (chaos-hyperchaos) intermittency the largest transverse Lyapunov exponent is positive but small. Due to the finite time fluctuations of it there are stretches of time where the orbit is attracted to the invariant manifold M (i.e. the fluctuations may permit all transient Lyapunov exponents negative in these stretches). For $D < D_0$ the largest transverse Lyapunov exponent is sufficiently large and an escape to a completely different attractor is possible. In this case the chaotic attractor A becomes a normally repelling chaotic saddle, i.e. if there is an attractor in the invariant manifold M, but all points which are not lying on this manifold eventually leave a neighbourhood of A.

The typical bifurcations of system (3) are summarized in Fig. 1. Each of these bifurcations can be seen as supercritical (subcritical) according to the creation of nearby invariant sets as D increases (decreases) through the bifurcation points. In Fig. 1 bifurcations are shown as subcritical.

3. TRANSVERSE LINEARIZATION IN THE NEIGHBOURHOOD OF A CHAOTIC ATTRACTOR

Let us introduce the transverse linearization of eqns (3a) and (3b) in the neighbourhood of the chaotic attractor A, i.e. in the neighbourhood of the fixed point $e(t) = \mathbf{0}$ with the condition $R(t) = x(t) = y(t) \in A$. In this case one obtains the equations

$$\dot{x} = h(x, 0) = f(x)$$
 (5)

$$\dot{e} = B(x; D)e, \tag{6}$$

where

$$B(x; D) = \frac{\delta g(R, e; D)}{\delta e} \bigg|_{e=0}^{R=x \in A}$$
(7)

The concept of transverse linearization in the neighbourhood of the attractor that we introduce here, allows us to reduce the stability analysis of the attractor to the problem of investigating the stability of the fixed point e(t) = 0 of the non-autonomous eqn (6). We emphasize that the origin e(t) = 0 in the Euclidean subspace defined by the vector e(t) represents the chaotic attractor A, while e(t) represents transverse perturbations in the neighbourhood of it. Fortunately, the linearized eqn (5), describing the goal attractor A is now decoupled from the transverse motion and independent of the parameter D. However, the transverse dynamics of eqn (6) is still coupled to the motion x(t) of the attractor through the matrix B(x; D).

We can state the following conjecture.

Conjecture. The chaotic attractor A of the n-dimensional dynamical system $\dot{x} = f(x)$ is asymptotically and monotonically stable in the 2n-dimensional phase space of the augmented dynamical system $\dot{X} = F(X)$, if and only if, for all $x \in A$, e(t) = 0 is an asymptotically stable fixed point of eqn (6).

In the following section we propose a simplified approach to predict the synchronization stability from eqns (5) and (6). Instead of calculating the *D*-dependent Lyapunov exponents of the chaotically forced transverse perturbations e(t), we merely investigate the local adiabatic stability of e(t) along the attractor. Such an analysis gives a quick idea of the locally stable and unstable parts of the attractor, but the success of the numerical predictions depends on the requirement that the evolution on the chaotic attractor is sufficiently smooth, and that there are not too many regions of different local stability types for the transverse perturbations. In this analysis the local eigenvalues of B(x; D) are the key quantities.

4. NUMERICAL EXAMPLE

As an example let us consider two Rossler systems, coupled in the same way as shown in eqn (1). We have the vector field

$$f(x) = \begin{pmatrix} -x_2 - x_3 \\ x_1 + ax_2 \\ b + (x_1 - c)x_3 \end{pmatrix},$$
(8)

where a, b and c are constants, and f(y) has the identical structure. When introducing the new variables e(t) and R(t), the original equation (1) is transformed into the equivalent one (3), which in this case has the explicit form:

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$$\dot{R} = \begin{pmatrix} -R_2 - R_3 \\ R_1 + aR_2 \\ b + (R_1 - c)R_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ e_1e_3 \end{pmatrix}$$
(9a)

$$\dot{e} = \begin{pmatrix} -2De_1 - e_2 - e_3\\ e_1 + (a - 2D)e_2\\ R_3e_1 + (R_1 - c - 2D)e_3 \end{pmatrix}.$$
 (9b)

Since the components of the transverse motion appear quadratically in (9a) they will disappear completely in the transverse linearization, and R(t) will describe the synchronized chaotic Rossler attractor, i.e. $R(t) = x(t) = y(t) \in A$. Equation (9b) is already linear in the components of e(t), so the linearized equations (5) and (6) are now defined with the matrix B(x; D) given by

$$B(x_1, x_3; D) = \begin{pmatrix} -2D & -1 & -1 \\ 1 & (a-2D) & 0 \\ x_3 & 0 & (x_1-2D-c) \end{pmatrix}.$$
 (10)

In our numerical investigation we consider the following parameter values: a = 0.15, b = 0.20, and c = 10.0. In the case of D = 0 (no coupling) the dynamics of both Rossler systems evolve along the chaotic attractor A [22]. This chaotic attractor is characterized by the following spectrum of Lyapunov exponents: $\lambda_1^{(1)} = 0.13$, $\lambda_2^{(1)} = 0$, $\lambda_3^{(1)} = -14.1$.

The exact system of equations (9) allows a direct numerical analysis of the transverse behaviour. In Fig. 2(a)-(d) we illustrate the typical evolution of the transverse flow



D=3.5

Fig. 2(a). Caption on p. 1576.

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Fig. 2(b). Caption on p. 1576.

corresponding to each characteristic behaviour indicated in Fig. 1. The top part of the figures shows the 3D-trajectories e(t) and the bottom part shows the time series of the vertical component $e_3(t)$. In Fig. 2(a) the chaotic attractor A represents a monotonically asymptotically stable one, and we can observe that the distance between points on the transverse trajectory e(t) and the attractor A seems to decrease monotonically. Asymptotic stability which is not monotonic is described in Fig. 2(b). Transverse evolution e(t) approaches the attractor A and finally finishes on it, but its distance from the attractor may sometimes increase slightly, as is visible in the $e_3(t)$ plot. In Fig. 2(c) we observe an on-off (chaos-hyperchaos) intermittency. The transverse trajectory e(t) spends some time in the neighbourhood of the chaotic attractor, but occasionally bursts away from it. Finally in Fig. 2(d) the periods of evolution in the neighbourhood of A are not visible any more and the chaotic attractor is a normally repelling chaotic saddle.

For this example the transverse linearized flow in the neighbourhood of the chaotic attractor can be given an adiabatic interpretation of its local stability, parametrized by the x_1 - and x_3 -components of the attractor. In Fig. 3(a)-(d) we show the characteristic types of eigenvalues of the matrix $B(x_1, x_3; D)$ for x_1 and x_3 in the ranges [-20.0 20.0] and [-1.0 40.0], respectively. In the grey regions all eigenvalues are either real and negative or complex with negative real parts, while in the white regions at least one real eigenvalue is positive or a pair of complex eigenvalues have positive real parts. Figure 3(a) illustrates





Fig. 2(c). Caption overleaf.

that, for $D = D_3 = 3.5$ all $(x_1, x_3) \in A$ are in grey region and the eigenvalues of the matrix $B(x_1, x_3; D)$ have all negative real parts, i.e., the fixed point e(t) = 0 is expected to be locally stable in the whole attractor, and so we conclude that the chaotic attractor A is monotonically stable.

For smaller values of D, part of the attractor is in the grey region and part of it is in the white one as shown in Fig. 3(b), (c). This means that in one part of the attractor e(t) = 0 is a locally stable fixed point, while in the other part it is unstable. From the adiabatic point of view it would be possible to investigate the asymptotic stability further by integrating the leading, local exponent of the transverse perturbation along with the attractor and see if the time averaged value stabilizes as negative or not. However, such a calculation would be similar to analysing Lyapunov exponents numerically. In this study we have confirmed in all our calculations that $D_3 > D > D_2 = 0.065$ (= $\lambda_1^{(1)}/2$) stabilizes the synchronized attractor within 1500 time units, but not monotonically.

As the coupling D is made smaller still $D < D_1$, on-off (chaos-hyperchaos) intermittency occurs in the combined system depending on the amount of time the trajectory in the 'goal' attractor shares between the locally stable and unstable regions.

For very small D ($D < D_0$) most of the attractor is in the white region as can be seen in Fig. 3(d). Since the chaotic attractor spends most of its time very near the surface $x_3 = 0$, it





Fig. 2. Typical behaviours of the transverse motion: (a) monotonic, asymptotic stability; (b) asymptotic stability; (c) on-off intermittency; (d) hyperchaos.

is very important for the contraction of the transverse motion that this surface is locally stable. There is a critical value $D = D_1 = 0.0375$, at which the whole surface $x_3 = 0$ changes its stability. Hence, for $D < D_0$ we do not expect any overall stability of the chaotic 'goal' attractor, as it becomes a normally repelling chaotic saddle.

5. CONCLUSIONS

In this paper we have investigated the transverse stability of the chaotic attractor of an n-dimensional dynamical system in the 2n-dimension phase space of two identical coupled systems. We identified and described a 'new' transition in which the chaotic attractor A looses monotonic stability. This phase transition seems to be characteristic for identical systems which are coupled to synchronize.

The introduced concept of transverse linearization in the neighbourhood of a chaotic attractor representing a synchronized state of two identical subsystems, allows us to describe its stability in three-dimensional subspace transverse to the attractor. The linear approach allowed to determine coupling (control) parameters for which system (1) undergoes two important types of bifurcations. One in which the asymptotic monotonic stability is replaced by asymptotic stability and another one when we have the end of on-off (chaos-hyperchaos) intermittency. However, we were not able to determine blowout bifurcations by this approach.



Fig. 3(a). Caption on p. 1580.



Fig. 3(b). Caption on p. 1580.



Fig. 3(c). Caption overleaf.



Fig. 3. Illustration of the local stability (grey region $\approx all \lambda^{(2)} < 0$) of the transverse motion along different parts of the attractor: (a) monotonic, asymptotic stability; (b) asymptotic stability; (c) on-off intermittency; (d) hyperchaos.

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