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Monotonic Stability

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Abstract—We show a new type of synchronization in coupled chaotic systems which is characterized by a small synchronization time. We introduce the concept of monotonic stability for which all perturbations decay monotonically and describe transition from a monotonically stable to an asymptotically stable chaotic attractor. Copyright © 1996 Elsevier Science Ltd.

The problem of synchronization of chaotic systems can be understood as a problem of the stability of an n-dimensional chaotic attractor in an m-dimensional phase space ($m > n$). Let A be a chaotic attractor. The basin of attraction $\beta(A)$ is the set of points whose ω-limit set is contained in A. In Milnor's definition [1] of an attractor the basin of attraction need not include the neighbourhood of the attractor. For example, a riddle basin [2, 3] which has recently been found in practical physical systems [4, 5], has a positive Lebesgue measure but does not contain any neighbourhood of the attractor. Attractor A is an asymptotically stable attractor if it is Lyapunov stable [$\beta(A)$ has a positive Lebesgue measure] and $\beta(A)$ contains a neighbourhood of A.

In this paper we define the monotonic stability of an attractor as a special case of asymptotic stability and show that it is characteristic for coupled systems.

Consider a system

$$\dot{z} = f(z) \tag{1}$$

consisting of two coupled identical subsystems governed by

$$\begin{aligned} \dot{x} &= f(x) + D(y - x) \\ \dot{y} &= f(y) + D(x - y), \end{aligned} \tag{2}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $n \geqslant 3$, $D \in \mathbb{R}^+$. Assume that $\dot{x} = f(x)$ and $\dot{y} = f(y)$ have the asymptotically stable chaotic attractor A in invariant subspace $\mathbb{R}^n = \mathbf{N}$ given by the relation $x = y$.

As it was shown in Ref. [6] chaotic attractor A is asymptotically stable in $\mathbb{R}^{2n}$ if $D > D_1 = \lambda/2$, where λ is the largest Lyapunov exponent of the chaotic state. In this case the synchronized state $\boldsymbol{x}(t) = \boldsymbol{y}(t)$ is achieved for all initial conditions in the neighbourhood of A. The detailed description of the dynamics of system (1) for $D < \lambda/2$ can be found in Ref. [7] and will not be discussed here. Dynamical phenomena characteristic for this range of D values have been also described in Refs [2, 3, 8–13].

With a further increase of D ($D > D_2$) we can observe a special case of the asymptotic stability of attractor A. Let $\mathbf{z}_0 = [\mathbf{x}_0, \mathbf{y}_0]^T \in \beta(A)$ be the initial perturbation of the trajectory of system (1) and let us define the distance of the perturbed trajectory $\mathbf{z}(t)$ from the attractor A as

$$\mathrm{d}(z(t), A) = \sqrt{((x_1 - y_1)^2 + \ldots + (x_n - y_n)^2)}. \tag{3}$$

If $\mathrm{d}(z(t), A)$ is a monotonically decreasing function of time t then attractor A is monotonically stable. It should be noted here that monotonic stability depends on a metric $\mathrm{d}(z(t), A)$. In this sense monotonicity is a quantitative property of the attractor and may depend on the observables. For example, the linear system $(\dot{x}, \dot{y}) = (-ax + by, -ay - bx)$ with $a > 0$, $b > 0$ exhibits a monotonic decay of $d = x^2 + y^2$, but if we choose $d = cx^2 + y^2$ with $c \gg 1$ this decay will no longer be monotonic. In this example eigenvalues are complex (with negative real part) so the transition from monotonic to asymptotic stability is not connected with the well known linear analysis transition from spiral to node.

This observation allows us to add a new transition characterized by the loss of monotonic stability to the already known blowout (chaos–hyperchaos) and loss of asymptotic stability bifurcations [7] which are characteristic for system (1). As the transition point between monotonic and asymptotic stability varies with the choice of metric, this transition cannot be called bifurcation. Each of these bifurcations and monotonic transitions can be seen to be supercritical (subcritical) according to the creation of nearby invariant sets as D increases (decreases) through the bifurcation point.

For the simplified analysis of the stability of chaotic attractor A let us introduce a new variable:

$$e(t) = x(t) - y(t). \tag{4}$$

With this transformation one replaces system (1) with the equivalent system

$$\dot{x} = f(x) - De \tag{5a}$$

$$\dot{e} = f(x) - f(x - e) - 2De = g(x, e). \tag{5b}$$

The first equation (5a) describes evolution in the neighbourhood of n-dimensional invariant subspace N, while the second equation (5b) describes the evolution transverse to subspace N. The spectrum of Lyapunov exponents of eqn (5) can easily be divided into two subsets $\lambda^{(1)}$ associated with the evolution of $x(t)$ describing dynamics in subspace N while the other set $\lambda^{(2)}$ describes propagation of perturbation normal to N.

Let us linearize eqn (5b) in the neighbourhood of the attractor A, i.e. in the neighbourhood of the fixed point $e = 0$ with the condition $x \in A$. In this case one obtains the equation,

$$\dot{e} = B(x, D)e, \tag{6}$$

where

$$B(x, D) = \left.\frac{\delta g(\bar{x})}{\delta \bar{x}}\right|_{x \in A, e=0} \tag{7}$$

and $x = [x, e]^T$.

The introduced concept of linearization in the neighbourhood of the attractor allows us to reduce the problem of analysis of the stability of the attractor to the problem of investigation of the stability of the fixed point $e = 0$ of eqn (6). Matrix $\mathbf{B}(x, D)$ is defined at given values of $x(t)$ which represents solution of eqn (5a) so numerical integration of eqn (5a) is necessary to calculate $\mathbf{B}(x, D)$. Eigenvalues of $\mathbf{B}(x, D)$ can be considered only as functions of $x^{(i)}$, where $x^{(i)}$ is a discrete series such as $x^{(i)} \in x(t)$, i.e. solution $x(t)$ has been discretized. Eigenvalues of the time-dependent matrix $\mathbf{B}(x, D)$, after appropriate averaging over the whole attractor A, lead to the Lyapunov exponents. Without averaging the properties of eigenvalues of $\mathbf{B}(x, D)$ this allows us to distinguish between monotonic and asymptotic stability, giving the following result.

Proposition: The chaotic attractor A is asymptotically and monotonically stable in metric (3) in $\mathbb{R}^{2n}$ if for all $x \in A$, $e = 0$ is the asymptotically stable fixed point of eqn (6).

Proof of the above proposition is elementary (it is based on the fundamental results of the linear stability) so it will not be given here.

As an example let us consider two coupled Rossler systems

$$\begin{aligned} \dot{x} &= -(y + z) + D(u - x) \\ \dot{y} &= x + ay + D(v - y) \\ \dot{z} &= b + z(x - c) + D(w - z) \end{aligned} \tag{8a}$$

$$\begin{aligned} \dot{u} &= -(v + w) + D(x - u) \\ \dot{v} &= u + av + D(y - v) \\ \dot{w} &= b + w(u - c) + D(z - w), \end{aligned} \tag{8b}$$

where a, b and c are constant. In our numerical investigation we considered the following parameter values: $a = 0.15$, $b = 0.20$, $c = 10.0$, i.e. in the case of $D = 0$ (no coupling) the dynamics of both Rossler systems evolve along the chaotic attractor A [14] and typical trajectories are characterized by the following Lyapunov exponents: $\lambda_1 = 0.13$, $\lambda_2 = 0$, $\lambda_3 = -14.1$.

For this example the transverse linearized flow in the neighbourhood of the chaotic attractor is given by

$$\begin{aligned} \dot{e}_1 &= -2De_1 - e_2 - e_3 \\ \dot{e}_2 &= -2De_2 + e_1 + ae_2 \\ \dot{e}_3 &= -2De_3 + ze_1 + (x - c)e_3 \end{aligned} \tag{9}$$

and is linear in the e variable.

For this example the transverse linearized flow in the neighbourhood of the chaotic attractor A can give an adiabatic interpretation of its local stability, parametrized in the x and z components of the attractor. In Fig. 1(a,b) we show the characteristic types of eigenvalues of the matrix $\mathbf{B}(x, z, D)$ for x and z in the ranges $[-20.0\ 20.0]$ and $[-1.0\ 40.0]$, respectively. In the grey regions all eigenvalues are either real and negative or complex with negative real parts, while in the white regions at least one real eigenvalue is positive or a pair of complex eigenvalues has positive real parts. Figure 1(a) illustrates that, for $D = 3.5$ all $(x, z) \in A$ are in the grey region, and the eigenvalues of the matrix $\mathbf{B}(x, z, D)$ all have negative real parts, i.e. the fixed point $e(t) = 0$ is asymptotically stable for all points on the attractor A. We conclude that the chaotic attractor is monotonically stable. We identified the monotonic transition point for $D = D_2 = 3.49$.

For smaller values of D, part of the attractor is in the grey region and part of it is in the white one as shown in Fig. 1(b). This means that in one part of the attractor $e(t)$ is a locally stable fixed point, while in the other part it is unstable. From an adiabatic point of view it would be possible to investigate the asymptotic stability further by integrating the leading local exponent of the transverse perturbation along with the attractor and see if the time averaged value stabilizes as negative or not. However, such a calculation would be similar to the numerical analysis of Lyapunov exponents. In this study we have confirmed in all our calculations that $D_1 > D > 0.065$ $(= \lambda_1/2)$ stabilizes the synchronized attractor, but not monotonically. All our numerical computations have been performed using the software INSITE [15].

The stronger case of monotonic stability occurs when we observe that all components of $e(t)$ are eventually monotonic, i.e. when

$$|e_i| = \sqrt{((x_i - y_i)^2)} \tag{10}$$

is a monotonically decreasing function of time. This property is difficult to predict but has potential practical applications in communications [16].

To summarize we have defined a monotonic stability of a strange chaotic attractor and analysed the transition from asymptotic stability to monotonic stability. Associated with monotonic stability, monotonic chaos synchronization is characterized by short synchronization time and may have practical applications. The introduced concept of linearization in the neighbourhood of the attractor allows us to describe this transition in three-dimensional subspace transverse to the attractor.

(a)

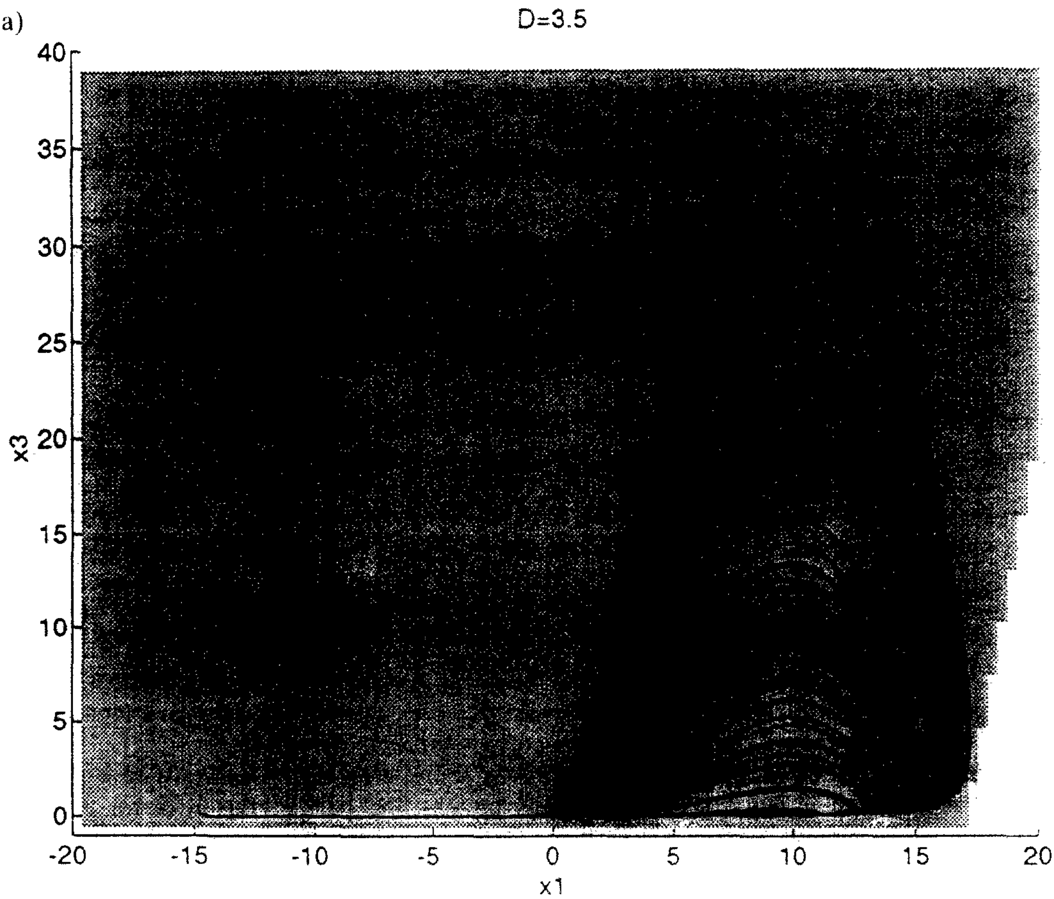

Fig. 1(a). *Caption opposite.*

(b)

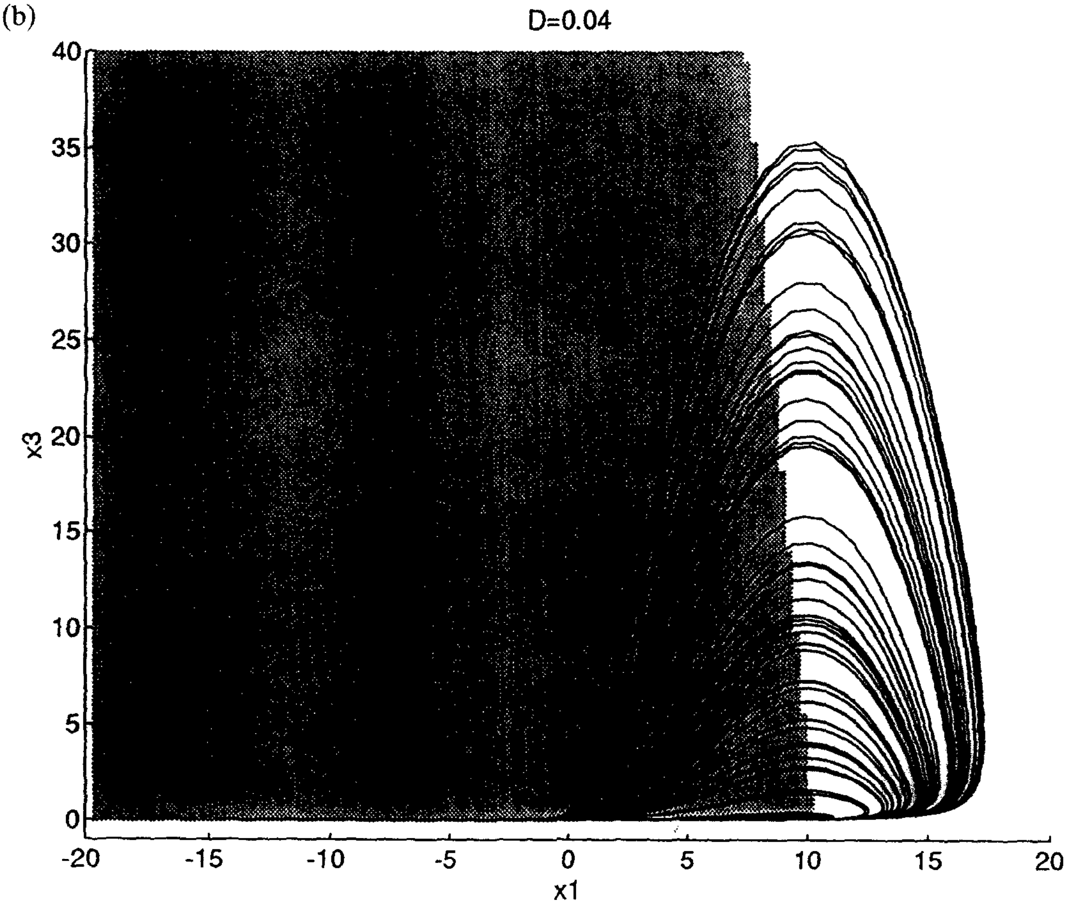

Fig. 1. Illustration of the local stability [grey region $e(t)$ locally asymptotically stable, white region $e(t) = 0$ locally unstable]: (a) monotonic, asymptotic stability; (b) asymptotic stability.

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