Uncertainty in coupled chaotic systems: Locally intermingled basins of attraction

Tomasz Kapitaniak
Division of Dynamics, Technical University of Łódź, Stefanowskiego 1/15, 90-924 Łódź, Poland
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We show that two coupled chaotic systems initially operating on two different simultaneusly coexisting attractors can be synchronized. Synchronization is achieved as one of the systems switches its evolution to the attractor of the other one. Final attractor of the synchronized state strongly depends on the actual position of trajectories on their attractors in the moment when coupling is introduced. Our system can be considered as an example of locally intermingled basins of attraction in physical system. [S1063-651X(96)01806-5]

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Synchronization of two chaotic systems has received considerable attention in the last few years [1–6]. Synchronization procedures require introduction of some kind of coupling between two chaotic systems. One of the synchronization procedures is based on the mutual coupling of two chaotic systems $\dot{\mathbf{x}}=f(\mathbf{x})$ and $\dot{\mathbf{y}}=f(\mathbf{y})$, $(\mathbf{x},\mathbf{y}\in\mathbf{R}^n,\ n\geqslant 3)$ by one-to-one negative feedback mechanism

$$\dot{\mathbf{x}}=f(\mathbf{x})+\mathbf{d}(\mathbf{y}-\mathbf{x}),$$

$$\dot{\mathbf{y}}=f(\mathbf{y})+\mathbf{d}(\mathbf{x}-\mathbf{y}), \tag{1}$$

where $\mathbf{d}=[d,d,\ldots,d]^T\in\mathbf{R}^n$ is a coupling vector. System (1) can be written as

$$\dot{\mathbf{z}}=g(\mathbf{z}), \tag{2}$$

where $\mathbf{z}=[x,y]^T\in\mathbf{R}^{2n}$. Manifold defined by the synchronized state $\mathbf{x}=\mathbf{y}$ is an invariant n-dimensional manifold of the system (2), i.e., any trajectory initialized in this manifold remains there for all time. This manifold is called a synchronization manifold.

The problem of synchronization of chaotic systems can be also understood as a problem of stability of n-dimensional chaotic attractor in m-dimensional phase space $(m>n)$. Let A be a chaotic attractor. The basin of attraction $\beta(A)$ is the set of points whose ω-limit set is contained in A. In Milnor's definition [7] of an attractor the basin of attraction need not include the neighborhood of the attractor. Attractor A is an asymptotically stable attractor if it is Lyapunov stable [$\beta(A)$ has positive Lebesque measure] and $\beta(A)$ contains neighborhood of A. Recently it has been shown that for certain types of systems the basin of attraction of attractor A can be riddled [8–14]. A riddle basin has a positive Lebesgue measure but does not contain any neighborhood of the attractor, i.e., for any point $\mathbf{x}_0$ in the riddled basin of an attractor a ball in the phase space of arbitrarily small radius $\mathbf{r}$ has a nonzero fraction of its volume in some other attractor's basin. The basin of the other attractor may or may not be riddled by the first basin. If the second basin is also riddled by the first one, we call such basins as intermingled. Riddled basins have been observer numerically and experimentally in a few physical systems [9–14]. Up to now intermingled basins have been observed in rather nonphysical maps [8].

Most of the work on the chaos synchronization problem has been associated with identical systems operating on some chaotic attractor. In this paper we investigate the dynamics of two coupled quasihyperbolic systems for which co-existing chaotic attractors are possible for the same parameter values. We examine the following questions: (1) can chaotic systems operating on different co-existing attractors synchronize?, (2) on which of the attractors synchronization occurs?

As an example we consider two Lorenz systems [15] coupled by one-to-one negative feedback mechanism,

$$\dot{x}_1=-\sigma x_1+\sigma y_1+d\,\theta(t-t_0)(x_2-x_1), \tag{3a}$$

$$\dot{y}_1=-x_1z_1+rx_1-y_1+d\,\theta(t-t_0)(y_2-y_1), \tag{3b}$$

$$\dot{z}_1=x_1y_1-bz_1+d\,\theta(t-t_0)(z_2-z_1), \tag{3c}$$

$$\dot{x}_2=-\sigma x_2+\sigma y_2+d\,\theta(t-t_0)(x_1-x_2) \tag{3d}$$

$$\dot{y}_2=-x_2z_2+rx_2+d\,\theta(t-t_0)(y_1-y_2), \tag{3e}$$

$$\dot{z}_2=x_2y_2-bz_2-+d\,\theta(t-t_0)(z_1-z_2), \tag{3f}$$

where σ, b, r and d are constants. $\Theta(t-t_0)$ is a Heaviside function [$\Theta(t-t_0)=1$ for $t\geqslant t_0$ and $\Theta(t-t_0)=0$ for $t<t_0$]. If both chaotic systems evolve on an asymptotically stable chaotic attractor A it can be shown that their evolutions would be synchronized for all t_0 when $d\geqslant\lambda/2$, where λ is the largest Lyapunov exponent of the typical trajectory on A.

In what follows we assumed that for $d=0$ both chaotic systems evolve on different chaotic attractors. Such a situation takes place, for example for $\sigma=10$, $b=8/3$ and $r=211$. Depending on initial conditions two coexisting attractors are possible. The first system [Eqs. (3a)–(3c)] is assumed to evolve on the attractor A_1 shown in Fig. 1(a) and the second one [Eqs. (3d)–(3f)] evolves on the attractor A_2 shown in Fig. 1(b). Both chaotic systems evolve on different attractors when coupling is introduced for $t=t_0$. For $t>t_0$ after the transition period the evolution of both systems is synchronized on one of the coexisting attractors A_1 or A_2. Such an evolution is shown in Fig. 2. During the transitional period the evolution of the first system [Eqs. (3a)–(3c)] $\mathbf{X}_1(t)(\mathbf{X}_1=[x_1,y_1,z_1]^T)$ was switched from attractor A_1 to A_2 then synchronized with the evolution of the second one [Eqs. (3d)–(3f)] $\mathbf{X}_2(t)(\mathbf{X}_2=[x_2,y_2,z_2]^T)$ on the attractor A_2.

The switch between attractor A_1 and A_2 (or A_2 and A_1) of the evolution of one of the systems is possible as the pertur-

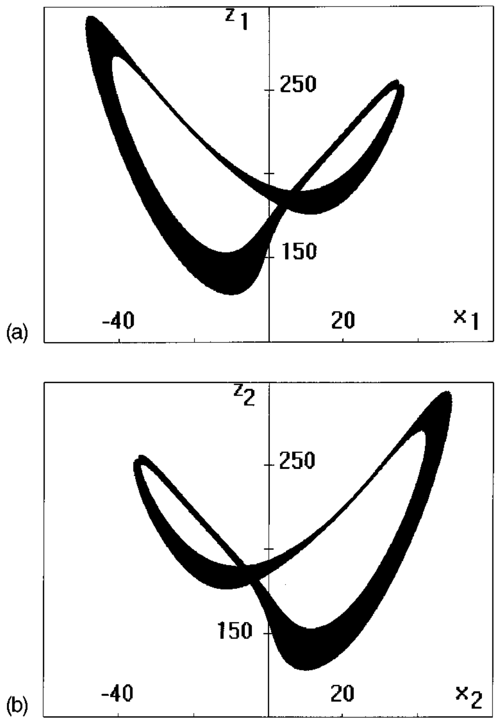

FIG. 1. Two co-existing attractors of uncoupled Eqs. (2): $d=0$, $\sigma=10$, $b=8/3$, and $r=211$; (a) $x_1(0)=21.0$, $y_1(0)=2.0$, $z_1(0)=0$, (b) $x_2(0)=-21.0$, $y_2(0)=0$, $z_2(0)=0$.

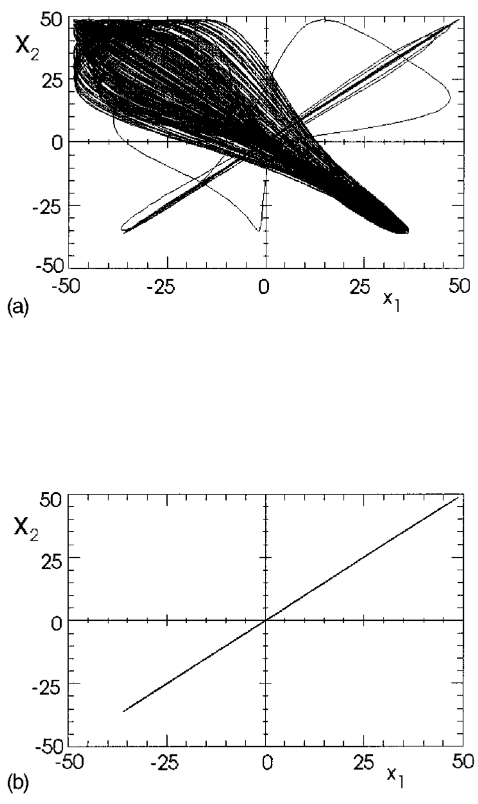

FIG. 2. Evolution of coupled Eqs. (2): $d=2$, $\sigma=10$, $b=8/3$, $r=211$, and $t_0=10^4$ (initially both systems evolve on attractors shown in Fig. 1).

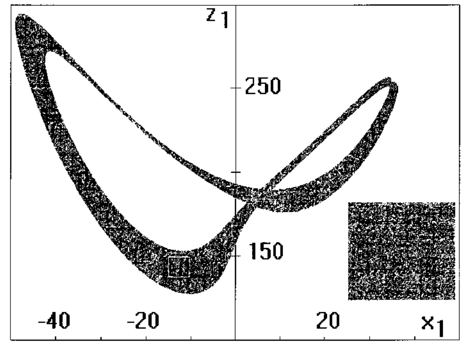

FIG. 3. Locally intermingled basins of attraction. The basins of attraction of attractors A_1 and A_2 are indicated, respectively, in white and black. In the computations shown in this figure initial location of the trajectory $X_2(t)$ on attractor A_2 have been fixed in the point $X_2(t_0)=[48.1422,44.6099,287.1101]^T$ and the location of the trajectory $X_1(t)$ on attractor A_1 have been varied.

bation $d(X_2-X_1)$ [or $d(X_1-X_2)$] moves trajectory $X_1(t)$ [or $X_2(t)$] out of the basin of attraction $b(A_1)$ of attractor A_1 [or $b(A_2)$ of attractor A_2] to the basin of attraction $b(A_2)$ of attractor A_2 [or $b(A_1)$ of attractor A_1]. Simultaneously the perturbation $d(X_1-X_2)$ [or $d(X_2-X_1)$] cannot move the trajectory $X_2(t)$ [or $X_1(t)$] out of the basin of attraction $b(A_2)$ of attractor A_2 [or $b(A_1)$ of attractor A_1]. Perturbed trajectory $X_2(t)$ [or $X_1(t)$] leaves attractor A_2 (or A_1) but evolves within its basin of attraction. Before the final switch to one of the attractors A_1 or A_2 both trajectories can switch several times between attractors in transitional period. Theoretically these switches could be permanent giving rise to the intermittent type of behavior, but we did not observe such an evolution in system (2).

We observed that the final attractor of the synchronized state strongly depends on the value of t_0 (on the time when coupling is introduced), i.e., on the initial locations $\mathbf{X}_{1,2}(t_0)$ ($\mathbf{X}_{1,2}=[x_{1,2},y_{1,2},z_{1,2}]^T$) of trajectories on attractors A_1 and A_2. At the time $t=t_0$ chaotic trajectories $\mathbf{X}_{1,2}(t)$ ($\mathbf{X}_{1,2}=[x_{1,2},y_{1,2},z_{1,2}]^T$) are at the points $\mathbf{X}_1(t_0)\in A_1$ and $\mathbf{X}_2(t_0)\in A_2$ which strongly depend on initial conditions $\mathbf{X}_1(0)$ and $\mathbf{X}_2(0)$ characterizing trajectories of both systems, introducing coupling at $t=t_0$ we are unable to predict on which attractor the synchronization occurs. $\mathbf{Z}(0)=[\mathbf{X}_1(0),\mathbf{X}_2(0)]^T$ can be considered as the initial conditions for the augmented $2n$-dimensional system (2). We performed our computations for 10^4 on randomly chosen initial conditions $\mathbf{X}_1(0)=[21.0\pm0.1,2.0\pm0.1,0\pm0.1]^T$ and $\mathbf{X}_2(0)=[-21.0\pm0.1,0\pm0.1,0\pm0.1]^T$ and introduced coupling at $t_0=10^4$ when both systems are on their attractors. Our results show that both chaotic attractors A_1 and A_2 are equally probable as a place of a synchronized regime. The basins of the attractors A_1,A_2 of the coupled system (3) [considered as a six dimensional system of the type (2)] on the chaotic attractors A_1 and A_2 of the uncoupled systems (3a)–(3c) and (3d)–(3f) are intermingled (Fig. 3). The basins of attraction of attractors A_1, and A_2 are indicated, respectively, in white and black. In the computations shown in Fig. 3 initial location of the trajectory $X_2(t)$ on attractor A_2 have been fixed in the point $X_2(t_0)=[48.1422,44.6099,187.1101]^T$ and the location of the trajectory $X_1(t)$ on attractor A_1 have been varied. Basins of attraction of the attractors A_1 and A_2 are not intermingled in whole six-dimensional phase space of system (3) [in the six-dimensional phase space the basins $b(A_1)$ and $b(A_2)$ have only fractal boundary], but are intermingled in the lower three-dimensional manifolds on which attractors A_1 and A_2 are located. As coupling in system (3) is introduced when both subsystems (3a)–(3c) and (3d)–(3f) are either on A_1 or A_2, all initial conditions for six-dimensional system (3) are located on a three-dimensional manifold where basins $b(A_1)$ and $b(A_2)$ are intermingled. This special case of basins of attraction which are intermingled on some lower-dimensional submanifold of the phase space but are not intermingled in the whole phase space we called locally intermingled. The phenomenon of locally intermingled basins of attraction seems to be typical for the systems with lower-dimensional synchronization manifold like system (2). All our numerical computations have been carried out using the software INSITE [16].

To summarize we showed here that in some quasihyperbolic systems synchronization of chaotic trajectories which initially evolve on different coexisting attractors is possible. The full discussion of this problem is given in [17]. One-to-one coupling introduced in such systems can lead to the locally intermingled basins of attraction of the initial attractors. Even if the initial location of trajectories on attractors A_1 and A_2 is known with infinite precision, we are unable to determine, on the basis of any finite calculation, in which basin this location lies and finally we cannot be sure on which attractor the evolution will synchronize. This type of uncertainty seems to be common for this class of dynamical systems with an invariant lower-dimensional manifold (synchronization manifold) and may have some practical implications.

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