15 January 1996

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 210 (1996) 283-289

# Synchronization and desynchronization in quasi-hyperbolic chaotic systems

Tomasz Kapitaniak [a], Jerzy Wojewoda [a], John Brindley [b]

[a] Division of Dynamics, Technical University of Lodz, Stefanowskiego 1 / 15, 90-924 Lodz, Poland
[b] Department of Applied Mathematical Studies, University of Leeds, Leeds LS2 9JT, UK

Received 21 June 1995; revised manuscript received 26 September 1995; accepted for publication 27 October 1995
Communicated by C.R. Doering

**Abstract**

We describe the synchronization and desynchronization procedures for chaotic orbits of quasi-hyperbolic systems.

## 1. Introduction

The synchronization of two nonlinear and potentially chaotic system is a phenomenon of some importance in a range of contexts varying from mechanical and electrical systems, where *control* of the system is the objective, to geophysical systems, like the atmosphere or oceans, where improvement in basic understanding and *prediction* is the main motivation. In particular it has been shown recently that two identical dynamical systems

$$\dot{p}=f(p, p_0, a) \tag{1}$$

and

$$\dot{q}=f(q, q_0, a), \tag{2}$$

where $p, q \in \mathbb{R}^n$, $n \geqslant 3$, $a \in \mathbb{R}$ is a control parameter, $p_0$ and $q_0$ represent initial conditions which for $a = a_1$ have an asymptotically stable chaotic attractor B (the same for both systems), can synchronize [1-9].

Synchronization may be achieved through a control procedure based on the OGY method [10-12], or through some form of coupling of the systems, and a common procedure is to introduce *small negative feedback*, in which the difference between the current states of the two systems is used as an inhibitory effect on the separation of trajectories. This method has been developed recently [13-16] and has been shown to be effective when the coupled system has a single attractor. In this paper we extend the idea to systems with more than one competing attractor, consider mechanisms for desynchronization, and apply the method to two different chaotic systems.

Consider the synchronization procedure developed in Refs. [13-16] applied to the system (1), (2). We assume that the second chaotic system is coupled with the first one by negative feedback,

$$g(p, q) = d(p-q), \tag{3}$$

where $d=[d_1, d_2, \dots, d_n]^T$, $d_i > 0$, $i = 1, 2, \dots, n$, is a coupling constant. Coupling (3) results in the following dynamical system,

$$\dot{p}=f(p, a), \quad \dot{q}=f(q, a) + d(p-q). \tag{4}$$

As it was shown in Refs. [7-9] it is possible to find such values of $d$ that both chaotic systems synchro-


SSDI 0375-9601(95)00880-2

nize for all initial points $p_0$ and $q_0$ in the neighbourhood of chaotic attractor B, i.e. $p(t) = q(t)$.

The above results hold if B is the only possible asymptotically stable attractor of systems (1) and (2) for $a_1$. However, when Eqs. (1) and (2) describe a quasi-hyperbolic system with at least two co-existing attractors, the synchronization procedure is not straightforward.

If the trajectory of one system is on the attractor $A_1$ and the trajectory of the other one is on the co-existing attractor $A_2$, to achieve synchronization, one of the trajectories, let us say the one on the attractor $A_1$, has to be perturbed in such a way that it goes to the basin of attraction $b(A_2)$ of the other attractor $A_2$. Let $e(A_1)$ be the region of the phase space in which the perturbed trajectory $x(t)$ evolves. The necessary condition for synchronization can be given by

$$e(A_1) \cap b(A_2) \neq \emptyset \tag{5}$$

(see Fig. 1). In some cases to fulfill relation (5) a strong perturbation which could be difficult to realize practically is necessary.

In this Letter we discuss the problems of synchronization and desynchronization of two quasi-hyperbolic systems using procedure (4) with a small negative feedback. Section 2 describes an example where relation (5) is fulfilled and two chaotic systems can be directly synchronized. In Section 3 we introduce the method which under additional conditions allows synchronization even when relation (5) is not initially fulfilled. Finally in Section 4 we summarize our results.

All numerical computations have been carried out using the software INSITE [17].

## 2. Direct synchronization

In a geophysical context, the phenomenon of synchronization is reminiscent of blocking features, in which the atmosphere enters into somewhat anomalous states which can prove remarkably persistent. These blocking episodes can occur when baroclinic systems in different longitudes become locked in some nearly synchronous mode of behaviour. The Lorenz system of equations has sometimes been proposed as a paradigm for the "chaotic" extratropical circulation [18]. Coupling between two Lorenz systems might then be interpreted as mutual interaction between extratropical circulation patterns in two different geographical regions having an essential control parameter which may be the same or have different values in the two regions.

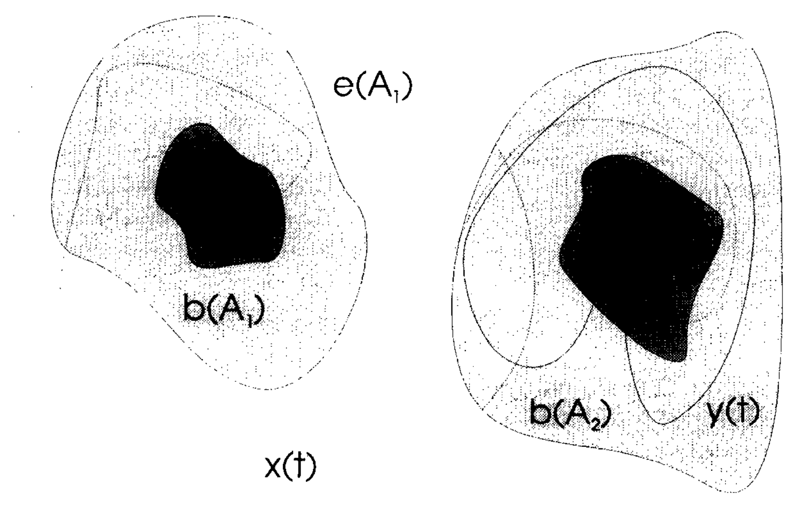


Fig. 1. Necessary condition for synchronization of two chaotic systems.

The concept of teleconnections of this kind, achieved though the mechanism of quasilinear Rossby wave trains, has both theoretical and experimental support [19–22].

We therefore consider a coupled pair of Lorenz systems, given by the equations

$$\dot{X}_1 = -\sigma X_1 + \sigma Y_1, \tag{6a}$$

$$\dot{Y}_1 = -X_1 Z_1 + r_1 X_1 - Y_1, \tag{6b}$$

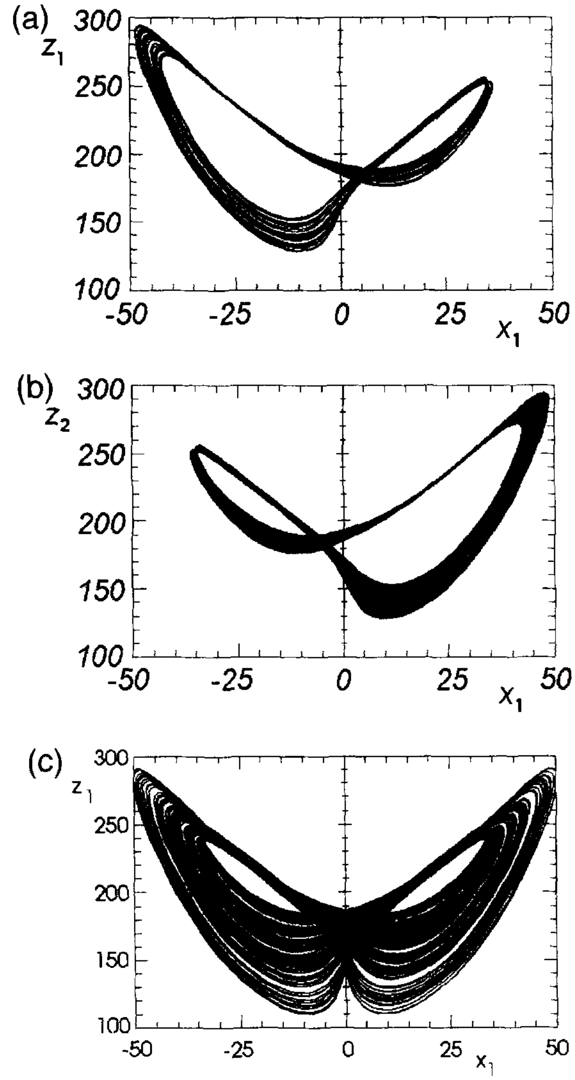


Fig. 2. Attractors of uncoupled Eqs. (6): $d = 0$, $\sigma = 10$, $b = 8/3$; (a), (b) two co-existing attractors for $r = 211$; (a) $x_1(0) = 21.0$, $y_1(0) = 2.0$, $z_1(0) = 0$, (b) $x_2(0) = -21.0$, $y_2(0) = 0$, $z_2(0) = 0$; (c) symetrical attractor for $r = 219$.

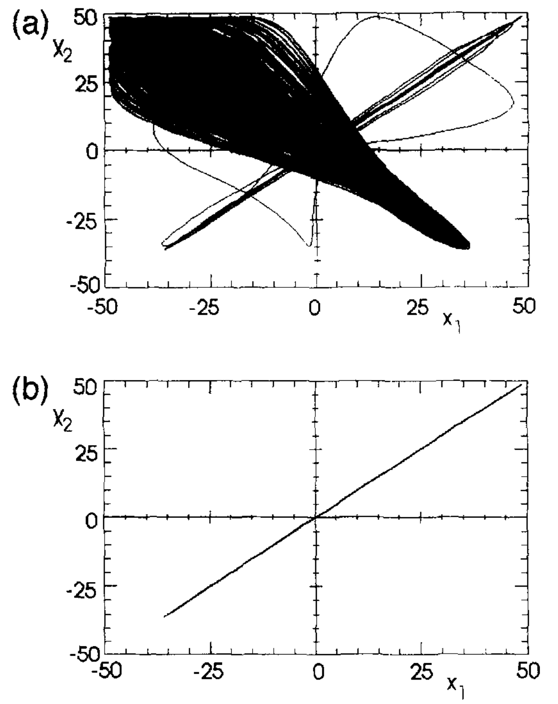


Fig. 3. (a) Evolution towards synchronized state of coupled Eqs. (6): $d_{1-3} = 2$, $\sigma = 10$, $b = 8/3$, $r = 211$ (initially both systems evolve on attractors shown in Fig. 2); (b) final synchronized state.

$$\dot{Z}_1 = X_1 Y_1 - bZ_1, \tag{6c}$$

$$\dot{X}_2 = -\sigma X_2 + \sigma Y_2 + d(X_1 - X_2), \tag{6d}$$

$$\dot{Y}_2 = -X_2 Z_2 + r_2 X_2 + d(Y_1 - Y_2), \tag{6e}$$

$$\dot{Z}_2 = X_2 Y_2 - bZ_2 + d(Z_1 - Z_2), \tag{6f}$$

where $\sigma$, $r_{1,2}$ and $b$ are constants. All state variables of both systems are coupled linearly with equal coupling strength $d$; the parameters $\sigma$ and $b$ are held fixed at $\sigma = 10.0$, $b = 8/3$, and $r_1$, $r_2$ are used as control parameters.

For certain ranges of $r_i$, each individual system can be on one of two attractors, mirror images of each other (Figs. 2a, 2b); for other ranges of $r_i$ only a single symmetric (butterfly) attractor exists (Fig. 2c).

Choosing $r_1 = r_2 = 211.0$, we have such a situation, and without coupling (i.e. $d = 0$) we can choose initial conditions so that system 1 (Eqs. (6a)–(6c)) is on atrractor $A_1$, say, whilst system 2 (Eqs. (6d)–(6f)) is on attractor $A_1$.

When we introduce coupling, even at a very weak level, we find that, in contrast to the further results

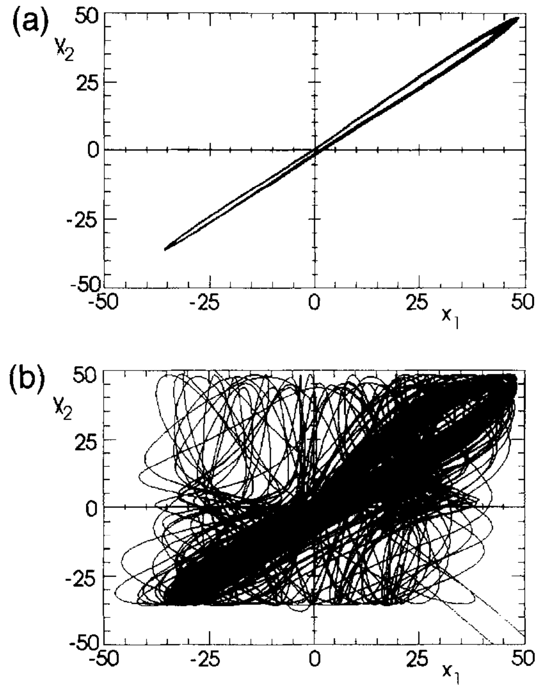


Fig. 4. (a) "Noisy synchronization" of Eqs. (6a)–(6c) and (6d)–(6f): $d_{1-3} = 2$, $\sigma = 10$, $b = 8/3$, $r_1 = 211$, $r_2 = 213$; (b) break of "noisy synchronization" $r_2 = 215$.

for the Chua equations of Section 3, synchronization occurs on attractor $A_1$ (Figs. 3a, 3b). In Fig. 3a the transient evolution towards the synchronized state of Fig. 3b is shown.

Thus synchronization is normal ($p = q$) in coupled Lorenz systems having *identical* values of $r$. When the values of $r_1$, $r_2$ are different, synchronization, by definition, cannot occur. However, as we show in Fig. 4a, for a range $r_1$, $r_2$, a "noisy synchronization" (i.e. $p \neq q$, but $\| x - y \| \leqslant \epsilon$ where $\epsilon$ is a vector of small parameters) occurs. Indeed there is a noisy modulation about synchrony (Fig. 3b) which persists in the case illustrated over the range, for $211.0 < r_2 < 215.0$. For larger values of $r_2$ the system evolves in the neighbourhood of a synchronized state for significantly long periods of time occacionally bursting out of this neighbourhood as can be seen in Fig. 4b. This final collapse of synchronization is associated with the replacement of $A_1$ and $A_2$ by a single symmetric attractor B (Fig. 2c).

These results inspire a number of speculations about the behaviour in geophysical fluids. Principal among these is the case in which synchronization is achieved in coupled Lorenz systems, its persistence and relative difficulty of desynchronization. This is perhaps significant in understanding the effectiveness of teleconnections, in which weak signals from distant features can apprarently have significant and sometimed dramatic effects on other weather systems.

## 3. Indirect synchronization

If for a different value of the control parameter $a$, let say $a = a_2$, Eqs. (1) and (2) have different chaotic or periodic attractors in different regions of the phase space, a small coupling (3) will not result in the synchronized state $p(t) = q(t)$ (relation (4) will not be fulfilled) and in the $p_i - q_i$, $i = 1, 2, \ldots, n$, plots we observed open curves or close Lissajou figures instead of a straight line.

In this section we describe a simple method which allows us to obtain synchronization of periodic and chaotic trajectories evolving on different co-existing attractors $A_1$ and $A_2$ which are close to the single chaotic attractor B.

Let us assume that the "one attractor" ($a_1$) and "co-existing attractors" ($a_2$) values of the control parameter $a$ are close together. Then the chaotic or periodic behaviour of Eqs. (1) and (2) can be synchronized through the following coupling,

$$\dot{p} = f(p, p_0, a(t)),$$
$$\dot{q} = f(q, q_0, a(t)) + K(p - q), \tag{7}$$

where

$$a(t) = a_1, \quad t \in [0, \tau_s],$$
$$= a_2, \quad t > \tau_s, \tag{8}$$

and $\tau_s$ is the synchronization time of chaotic systems (1) and (2) (the time in which chaotic systems (1) and (2) are synchronized). In the synchronization scheme (5) the value of $a$ is first fixed to the "one attractor" value $a_1$. When the synchronization state $p(t) = q(t)$ is achieved, parameter $a$ is switched to the "co-existing attractors" value $a_2$. The equality $p(\tau_s) = q(\tau_s)$ ensures the same initial conditions for the transient evolution towards one of the co-existing attractors so for $t > \tau_s$ we always have $p(t) = q(t)$ and the synchronization of trajectories is guaranteed.

As an example consider a pair of unidirectionally coupled identical Chua circuits whose combined equations of motion are

$$\dot{x} = \alpha(y - x - f(x)), \quad (9a)$$

$$\dot{y} = x - y + z, \quad (9b)$$

$$\dot{z} = -\beta y, \quad (9c)$$

$$\dot{u} = \alpha(v - u - f(u)), \quad (9d)$$

$$\dot{v} = u - v + w + d_2(y - v), \quad (9e)$$

$$\dot{w} = -\beta v, \quad (9f)$$

where

$$f(\zeta) = b\zeta + \tfrac{1}{2}(a - b)(|\zeta + 1| - |\zeta - 1|). \quad (10)$$

$\zeta = x$, $u$ and $\alpha$, $\beta$, $a$, $b$ are constants. The second Chua circuit (Eqs. (9d)–(9f)) is coupled to the first one (Eqs. (9a)–(9c)) in such a way that the differences between the signals $y$ and $v$ are ($d = [0, d_2, 0]^T$) introduced into the second circuit as a negative feedback.

In our investigation we considered the following parameter values: $\beta = 14.87$, $a = -1.27$ and $b = -0.68$. In the case of $d_2 = 0$ (no coupling) and $\alpha_1 = 10.0$ the dynamics of both Chua circuits evolve along the double-scroll Chua attractor [14] while for $\alpha_2 = 8.0$ two co-existing Rössler type attractors are possible.

If we start with the value of $a$ fixed to the "co-existing attractors" value $a_2$ the two systems (1) and (2) evolve on the different attractors $p(t) \neq q(t)$, where $p(t) = [x(t), y(t), z(t)]^T$ and $q(t) = [u(t), v(t), w(t)]^T$ as can be seen for the periodic and chaotic case in Figs. 5a, 5b. When we start with "one attractor" value $a_1$ and after achieving synchronization of chaotic trajectories $a$ is switched to $a_2$ then trajectories $p(t)$ and $q(t)$ evolve on one attractor and are synchronized as can be observed in Figs. 6a, 6b. Fig. 6a presents the transient evolution from the initial state of Fig. 5b to the final synchronized state shown in Fig. 6b. In this simulation $a_1 = \alpha_1 = 10.0$ and $a_2 = \alpha_2 = 8.0$ were taken. Our numerical observations show that there is a hysteresis in the dynamical behaviour of the coupled system (4) as different behaviour is obtained when the control parameter is increased than when this parameter is decreased.

This poses the question whether or not it is possible to desynchronize the systems again, and by what

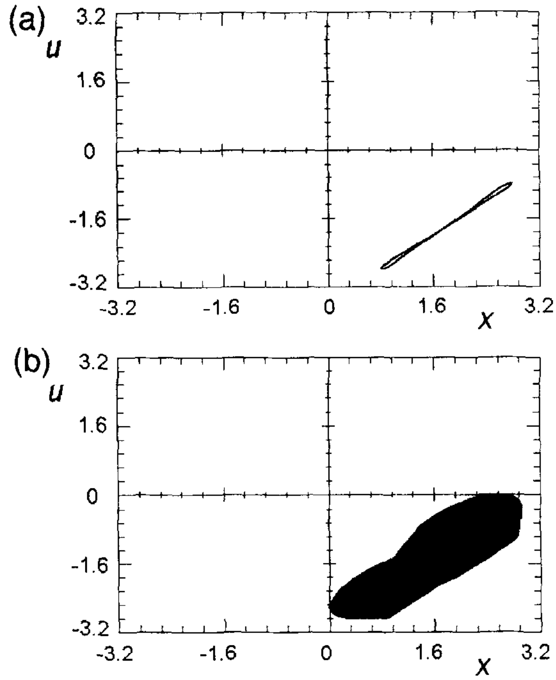


Fig. 5. Evolution of coupled Eqs. (9): $\beta = 14.87$, $a = -1.27$ and $b = -0.68$. In the case of $d_2 = 0$ (no coupling) both systems (Eqs. (9a)–(9c) and Eqs. (9d)–(9f)) evolve on different co-existing periodic or chaotic attractors; (a) Lissajou figure $\alpha = 7$; (b) unclosed curve $\alpha = 8$.

means. It is clear that any further parameter changes, in which the $a = \alpha$ values of both attractors change simultaneously, will fail to desynchronize the two systems. If, however, we switch the value of one of them, $a_1 = \alpha_1$ say, to the single attractor value, $a_1 = \alpha_1$, leaving the other at $a_2 = \alpha_2$, the two systems must have totally different attractors, B and $A_2$, say, and hence are desynchronized. When we switch $a = \alpha$ back, the second system switches either to attractor $A_1$ or $A_2$, according to its position on its trajectory at the time of switching (which must be in one of the basins of attraction $b(A_1)$ or $b(A_2)$ of $A_1$ or $A_2$). If it switches to $A_1$, synchronization occurs, if it switches to $A_2$, the systems remain desynchronized.

We summarise all possibilities in the diagram shown in Fig. 7. Thus, the probabilities of returning to the original value, $a_2 = \alpha_2$ in either a synchronized or desynchronized state will be proportional to the length of time the trajectory on B spends in the

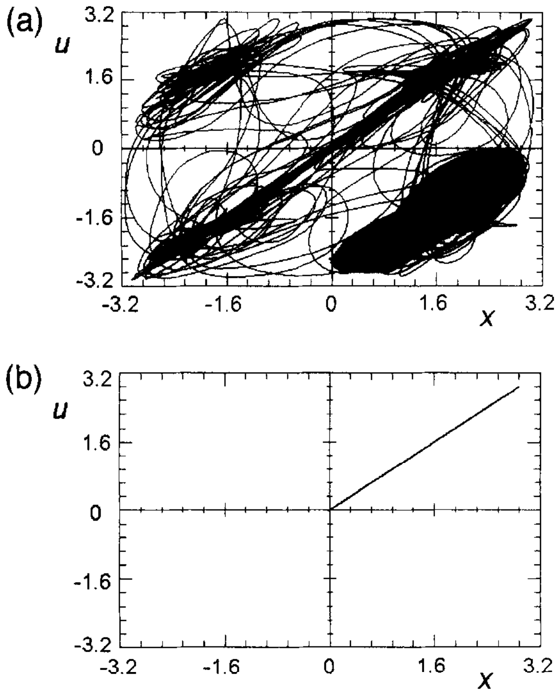


Fig. 6. Evolution from desynchronized state of Fig. 5b towards synchronization using scheme (7); $\alpha_1 = 10$, $\alpha_2 = 8$; (a) transient behaviour; (b) final synchronized state.

basins of $A_2$, $A_1$ respectively. For long trajectories this is the ratio

$$b(B) \cap b(A_2) : b(B) \cap b(A_1)$$

and it follows that, if $b(B) \cap b(A_1)$ is "small", the system is difficult to desynchronize, i.e. it will tend to persist in synchronized states.

## 4. Conclusions

Synchronization of chaotic and periodic orbits of coupled quasi-hyperbolic systems is straightforward to accomplish when the whole system has a single attractor. When the system has two or more attractors, synchronization may not be possible directly, but may be accomplished by shifting the value of a control variable temporarily to a value at which the system has just one attractor, synchronizing on that, and shifting the control variable back to its original value. This is a necessary procedure for the example of coupled Chua circuits examined in Section 3, but for the coupled Lorenz systems of Section 2, synchronization is achieved, even for very weak coupling but of all state variables, when the individual systems start on different attractors. Once synchronized, the coupled systems remain synchronized in each case, even for changes in the control parameter, as long as the control parameter remains the same for each of the coupled systems. Desynchronization can be achieved by decreasing the coupling stiffness $d$ in such a way that a synchronized state becomes unstable and adding an external perturbation to one

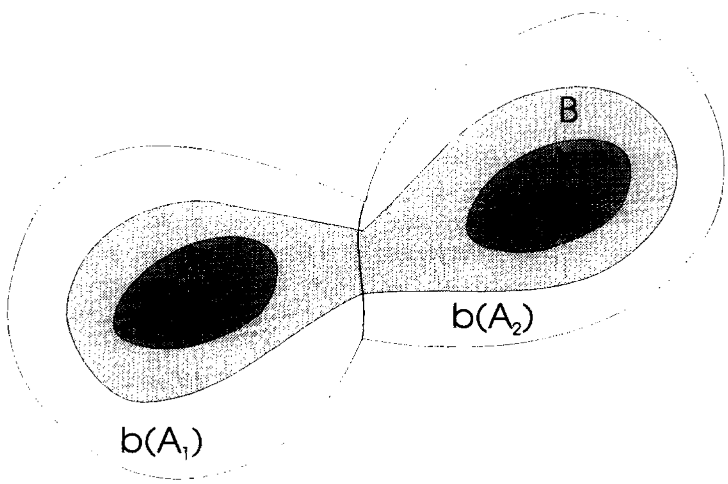


Fig. 7. Possibilities of desynchronization of Eqs. (9).

of the synchronized systems. In other cases desynchronization requires a change in the control parameters to values which are different in the two systems. For the Chua case, desynchronization when it occurs is immediately total. For the Lorenz system there is a range of values of the control parameter, terminated at a change in attractor topology, for which noisy synchronization persists. In practical application this may be as effective as, and quite difficult to distinguish from, genuine synchronization.

It should be mentioned here that the problem of controlling chaos in multiple attractor systems has been considered by Jackson and Hubler [23,24]. The method of migration control developed by them allows one to direct the system trajectory to the desired attractor. Alternatively to our approach synchronization of chaotic systems initially operating on different co-existing attractors can be achieved in two stages: (i) by initial migration control of the trajectory of one of the systems, (ii) by a standard synchronization scheme. Our approach gives a simpler method as initial control is not necessary.

## Acknowledgement

This work has been supported by The British Council/KBN under the project No. WAR/992/033.

## References

[1] T. Yamada and H. Fujusaka, Prog. Theor. Phys. 70 (1983) 1240.
[2] V.S. Afraimovich, N.N. Verichev and M.I. Rabinovich, Radiophys. Quantum Electron. 29 (1986) 795.
[3] L. Pecora and T.S. Carroll, Phys. Rev. Lett. 64 (1990) 821.
[4] L. Pecora and T.S. Carroll, IEEE Trans. Circuits Syst. 38 (1991) 453.
[5] V.S. Anishchenko, T.E. Vadivasova, D.E. Postnov and M.A. Safanova, Radioeng. Electron. 36 (1991) 338.
[6] K.M. Cuomo and A.V. Oppenheim, Phys. Rev. Lett. 71 (1993) 65.
[7] M. De Sousa, A.J. Lichtenberg and M.A. Lieberman, Phys. Rev. A 46 (1992) 7359.
[8] T. Endo and L.O. Chua, Int. J. Bif. Chaos 1 (1991) 701.
[9] U. Parlitz, L.O. Chua, L. Kocarev, K. Halle and A. Shang, Int. J. Bif. Chaos 2 (1992) 973.
[10] E. Ott, C. Grebogi and J.A. Yorke, Phys. Rev. Lett. 64 (1990) 1196.
[11] F.J. Romeiras, C. Grebogi, E. Ott and W. Dayawansa, Physica D 58 (1992) 165.
[12] Y-C. Lai and C. Grebogi, Phys. Rev. E. (1993) 2357.
[13] K. Pyragas, Phys. Lett. A. 170 (1992) 421.
[14] K. Pyragas, Phys. Lett. A 181 (1994) 203.
[15] T. Kapitaniak, Phys. Rev. E 50 (1994) 1642.
[16] T. Kapitaniak, L.O. Chua and G.-Q. Khong, Int. J. Bif. Chaos, 4 (1994) 483.
[17] T.S. Parker and L.O. Chua, Practical numerical algorithms for chaotic systems (Springer, Berlin, 1989).
[18] E.H. Lorenz, J. Atmos. Sci. 20 (1963) 130.
[19] R. Madden and P.R. Julian, J. Atmos. Sci. 28 (1971) 702.
[20] K. Weickmann, J. Geophys. Res. 96 (1991) 3187.
[21] B.J. Hoskins and B.J. Karoly, J. Atmos. Sci. 38 (1981) 1179.
[22] J.M. Wallace and D.S. Gutzler, Mon. Weather Rev. 109 (1981) 785.
[23] E.A. Jackson, Phys. Lett. A 151 (1990) 478.
[24] E.A. Jackson and A. Hubler, Physica D 44 (1990) 407.