ELSEVIER

Physica D 98 (1996) 594–598

PHYSICA D

Dynamics of coupled Lorenz systems and its geophysical implications

Andrzej Stefanski [a], Tomasz Kapitaniak [a,*], John Brindley [b]

[a] Division of Dynamics, Technical University of Łódź, Stefanowskiego 1/15, 90-924 Łódź, Poland

[b] Department of Applied Mathematical Studies, University of Leeds, Leeds LS2 9JT, UK

Abstract

Dynamics of two Lorenz systems coupled through the negative feedback mechanism working in different chaotic ranges in investigated. Possible implications of the observed phenomena like noisy synchronization and controlling chaos by chaos mechanism in geophysics are discussed.

PACS: 05.45.+b

Keywords: Chaos; Coupled systems; Predictability; Synchronization; Lorenz model

1. Introduction

Recently it has been demonstrated that two identical chaotic systems $\boldsymbol{x} = f(\boldsymbol{x})$ and $\boldsymbol{y} = f(\boldsymbol{y})$ ($\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, n \geq 3$) coupled with each other can be synchronized [1–6]. The coupling of homochaotic systems (i.e. systems given by the same set of ODEs but with different values of system parameters) can lead to the noisy (practical) synchronization (i.e. $\boldsymbol{x} \neq \boldsymbol{y}$, but $\| \boldsymbol{x} - \boldsymbol{y} \| \leq \epsilon$ where ϵ is a vector of small parameters) [7–9]. In such coupled systems we can also observe periodic locking [10] or a significant change of chaotic behaviour of one or both systems [11,12]. These phenomena are possible as the coupled systems become a new augmented system, which has its own dynamics.

In what follows we discuss some phenomena characteristic for coupled chaotic systems pointing out possible geophysical importance. We consider two monochaotic systems:

$$\dot{x} = f(a_1, \boldsymbol{x}) \tag{1}$$

and

$$\dot{y} = f(a_2, \boldsymbol{y}) \tag{2}$$

($\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, a_{1,2} \in \mathbb{R}, n \geq 3$) coupled with each other by negative feedback in such a way that the augmented system is as follows:

$$\dot{x} = f(a_1, x), \qquad \dot{y} = f(a_2, y) + \boldsymbol{d}(x - y) \tag{3}$$

where $\boldsymbol{d} = [d, d, \ldots, d]^{\mathrm{T}} \in \mathbb{R}^n$ is a coupling vector.

In a geophysical context, synchronization phenomenon is reminiscent of blocking features, in which the atmosphere enters into somewhat anomalous states which can prove remarkably persistent. These blocking episodes can occur when baroclinic systems in different longitudes become locked in some nearly synchronous mode of behaviour. The Lorenz system of equations has sometimes been proposed as a paradigm for the "chaotic" extratropical circulation [13,14]. The coupling between two Lorenz systems might then be interpreted as mutual interaction between extratropical

* Corresponding author. Tel.: (4842) 332231; fax: (4842) 365646; e-mail: tomaszka@lodzl.p.lodz.pl.

PII S0167-2789(96)00106-6

circulation patterns in two different geographical regions having an essential control parameter which may be the same or have different values in two regions.

The concept of teleconnections of this kind, achieved through the mechanism of quasi-linear Rossby wave trains, has both theoretical and experimental support [15–18]. We therefore consider a pair of monochaotic Lorenz systems coupled in such a way (3) that the augmented system is given by equations:

$$\dot{X}_1 = -\sigma X_1 + \sigma Y_1, \tag{4a}$$

$$\dot{Y}_1 = -X_1 Z_1 + r_1 X_1 - Y_1, \tag{4b}$$

$$\dot{Z}_1 = X_1 Y_1 - b Z_1, \tag{4c}$$

$$\dot{X}_2 = -\sigma X_2 + \sigma Y_2 + d(X_1 - X_2), \tag{4d}$$

$$\dot{Y}_2 = -X_2 Z_2 + r_2 X_2 + d(Y_1 - Y_2), \tag{4e}$$

$$\dot{Z}_2 = X_2 Y_2 - b Z_2 + d(Z_1 - Z_2), \tag{4f}$$

where σ, $r_{1,2}$ and b are constants. All state variables of both systems are coupled linearly with equal coupling strength d; parameters σ and b are held fixed at $\sigma = 10.0$, $b = \frac{8}{3}$, and r_1, r_2 are used as control parameters.

2. Noisy synchronization

In most of the studies on chaos synchronization two identical or monochaotic systems operating on the same attractor are considered [1–6]. In this section we describe the possibility of chaos synchronization in quasi-hyperbolic systems (i.e. system with different co-excisting attractors) where both chaotic systems operate on different attractors.

For certain ranges of r_i, each individual system can be on one of the two attractors, mirror images of each other (Fig. 1(a) and (b)); for other ranges of r_i only a single symmetric (butterfly) attractor exists (Fig. 1(c)). Choosing $r_1 = r_2 = 211.0$, we have such a situation, and without coupling (i.e. $d = 0$) we can choose initial conditions so that system (1) (Eqs. 4(a)–4(c)) is on attractor A_1 of Fig. 1(a), say, whilst system (2) (Eqs. 4(d)–4(f)) is on attractor A_2 of Fig. 1(b).

When we introduce coupling synchronization occurs on attractor A_1 (Fig. 2(a) and (b)) as via coupling (3) system (1) forces system (2). In Fig. 2(a) the transient evolution towards synchronized state of Fig. 2(b) is shown.

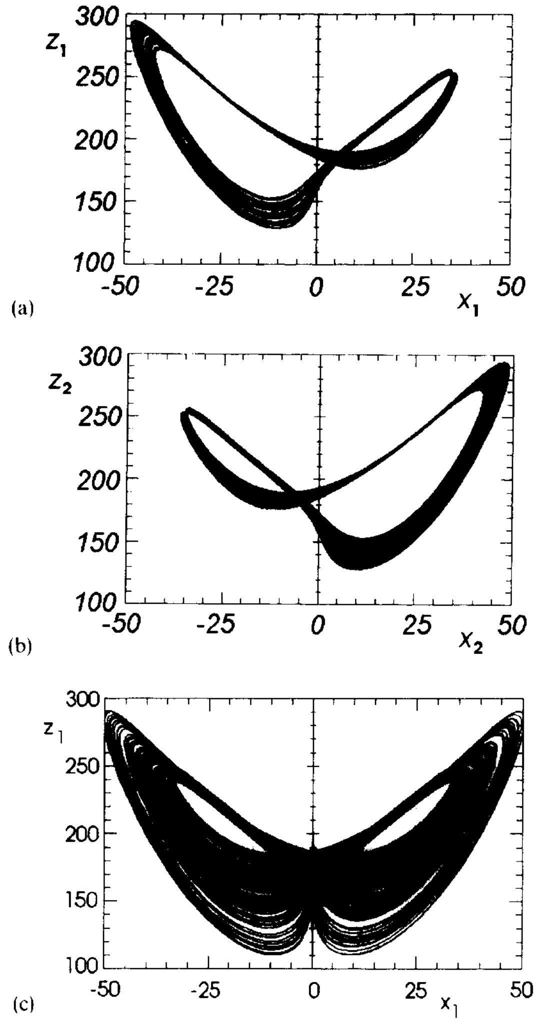

Fig. 1. Attractors of uncoupled Eqs. (4): $d = 0$, $\sigma = 10$, $b = \frac{8}{3}$; (a) $r = 211$, $x_1(0) = 21.0$, $y_1(0) = 2.0$, $z_1(0) = 0$, $x_2(0) = 0$, $y_2(0) = 0$, $z_2(0) = 0$; (b) $r = 211$, $x_1(0) = 0$, $y_1(0) = 0$, $z_1(0) = 0$, $x_2(0) = -21.0$, $y_2(0) = 0$, $z_2(0) = 0$, (c) $r = 197.4$.

Thus synchronization is normal ($\boldsymbol{x} = \boldsymbol{y}$) in coupled Lorenz systems having *identical* values of r. When the values of r_1, r_2 are different, synchronization, by definition, cannot occur. However, as we show in Fig. 3(a), for a range of r_1, r_2, "noisy synchronization" (i.e. $\boldsymbol{x} \neq \boldsymbol{y}$, but $\| \boldsymbol{x} - \boldsymbol{y} \| \leq \epsilon$ where ϵ is a vector of small parameters) takes place. Indeed there is a noisy modulation about synchrony (Fig. 2(b)) which persists in

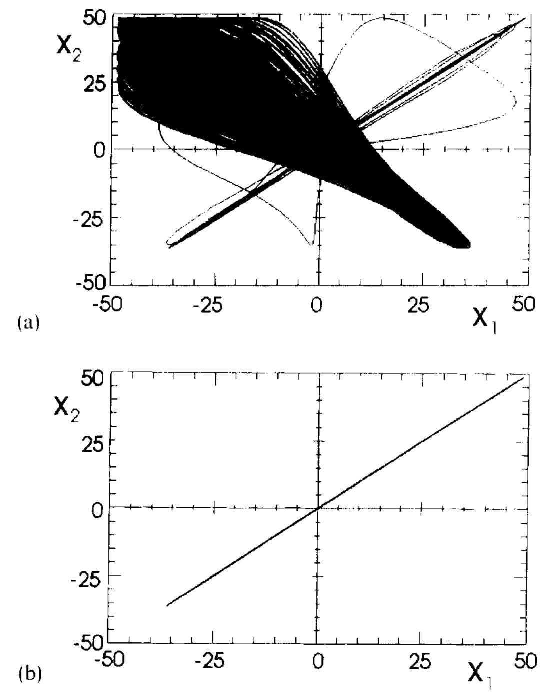

Fig. 2. (a) Evolution towards synchronized state of coupled Eqs. (4): $d = 2, \sigma = 10, b = \frac{8}{3}, r = 211$ (initially both systems evolve on attractors shown in Fig. 1(a) and (b)). (b) Final synchronized state.

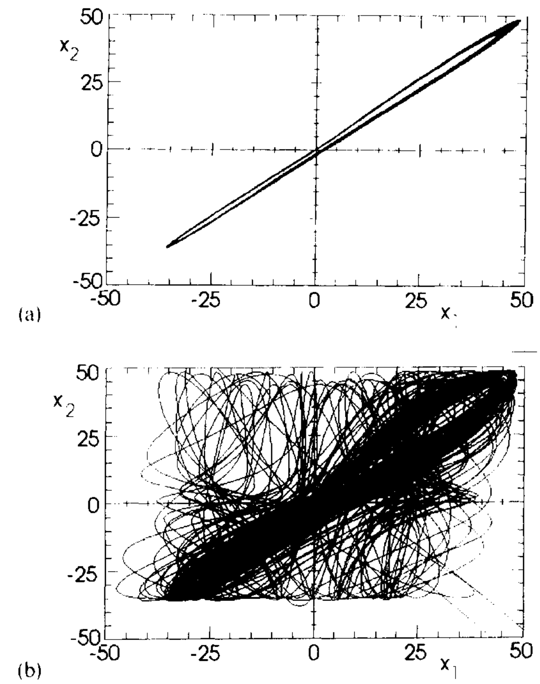

Fig. 3. (a) "Noisy synchronization" of Eqs. (4a)–(4c) and (4d)–(4f): $d = 2, \sigma = 10, b = \frac{8}{3}, r_1 = 211, r_2 = 213$. (b) Break of "noisy synchronization" $r_2 = 215$.

the case illustrated over the range for $211.0 < r_2 < 215.0$. For larger values of r_2 the systems evolve in the neighbourhood of synchronized state for a significantly long period of time occasionally bursting out of this neighbourhood as can be seen in Fig. 3(b). This final collapse of synchronization is associated with the replacement of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ by a single symmetric attractor $\boldsymbol{B}$ of a type shown in Fig. 1(c).

The difference between ideal synchronization defined by relation $\boldsymbol{x} = \boldsymbol{y}$ and noisy synchronization defined by relation $\| \boldsymbol{x} - \boldsymbol{y} \| \leq \epsilon$ where ϵ is a vector of small parameters can be described as follows. In the case of ideal synchronization the coupled system (1) and (2) evolve on the same manifold on which one of the chaotic systems (in our case system (1)) evolve (phase space is reduced to n-dimensional synchronization manifold $\boldsymbol{x} = \boldsymbol{y}$), while for noisy synchronization the coupled systems evolve on higher-dimensional manifold on which hyperchaotic attractor exists [6]. In the case of noisy synchronization the attractor of coupled systems (1) and (5) is hyperchaotic, systems' evolution takes place in the $2n$-dimensional phase space but it is restricted to the neighbourhood of n-dimensional synchronization manifold $\boldsymbol{x} = \boldsymbol{y}$.

Once synchronized or noisy synchronized, the coupled systems remain synchronized in each case, even for changes in the control parameter, so long as the control parameter remains the same for each of the coupled systems. Desynchronization can be achieved by decreasing coupling stiffness $\boldsymbol{d}$ in such a way that synchronized state becomes unstable and inputting external perturbation to one of the synchronized systems. In geophysical context this means that only local event (perturbation in one subsystem) can destroy synchronization in the global model (coupled system).

3. Enhanced predictability

In this section we show that even when the coupling (3) does not result in the systems, synchronization or noisy synchronization, it can modify the behaviour of one of the systems in useful way.

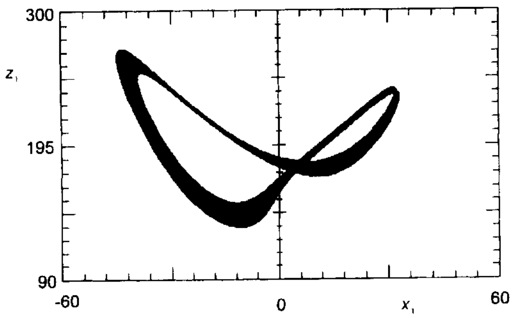

Fig. 4. Evolution of coupled Eqs. (4): $d = 2, \sigma = 10, b = \frac{8}{3}, r_1 = 211$ and $r_2 = 197.4$ (initially both systems evolve on attractors shown in Fig. 1(b) and (c)).

Let us recall Fig. 1(b) and (c). In these figures we show the chaotic attractors of single Lorenz models ($d = 0$) for $\sigma = 10.0, b = \frac{8}{3}$, and $r_1 = 211$ (Fig. 1(a)), $r_2 = 197.4$ (Fig. 1(b)). These attractors are characterized by the following spectra of Lyapunov exponents $\lambda_1 = 0.78, \lambda_2 = 0, \lambda_3 = -14.44$ (Fig. 1(a)) and $\lambda_1 = 1.87, \lambda_2 = 0, \lambda_3 = -15.54$ (Fig. 1(c)). In Fig. 4 we show the behaviour when the above Lorenz systems are unidirectionally coupled with $d = 100$. Although the final attractor is still chaotic ($\lambda_1 = 0.79, \lambda_2 = 0, \lambda_3 = -14.54$), trajectory behaviour on them is more predictable as its Lyapunov dimension $d_L = 2.044$ are smaller than the dimension of the original attractor of Fig. 1(c) ($d_L = 2.121$).

This dimension decrease is produced by a significant decrease of positive Lyapunov exponent ($\lambda_1 = 0.79$ in comparison with $\lambda_1 = 1.87$ of the original attractor). In Fig. 5 we show the plots of the measure of predictability $\kappa = 1/\lambda_{max}$, where λ_{max} is the largest Lyapunov exponent, versus coupling coefficient d. The analysis of Fig. 5 shows that for sufficiently large d the predictability is significantly increased.

This type of controlling chaos we called "controlling chaos by chaos" method and its theoretical bases are described in [11]. The application of coupling procedure (3) allowed us to convert one type of chaotic behaviour to chaotic behaviour which is more predictable when parameter d is close to the one that fulfils conditions given in [11]. For d which is far away from the conditions stated in [11] our controlling procedure does not work.

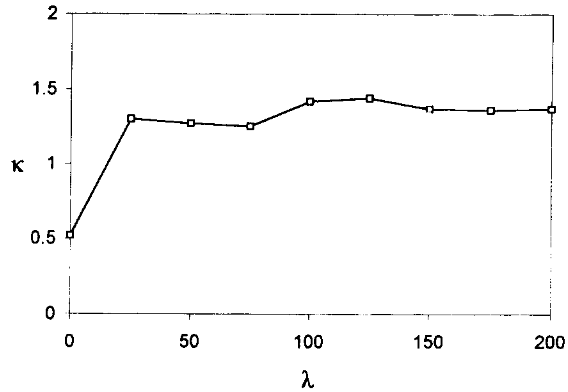

Fig. 5. Measure of predictability $\kappa = 1/\lambda_{max}$, where λ_{max} is the largest Lyapunov exponent, versus coupling coefficient d.

In cases considered in this paper the application of a control scheme did not result in obtaining periodic behaviour because the external periodic perturbation was not taken to coincide with an unstable periodic orbit of the original chaotic system. If that is done, a similar controlling scheme allows us to convert the original chaotic behaviour into an appropriate periodic one [4,6]. It should be mentioned here that, if $a_1 = a_2$ in Eqs. (1) and (2), our controlling procedure simplifies itself to the method of synchronization chaos using continuous control [6].

4. Conclusions

The results presented in this paper inspire a number of speculations about behaviour in the geophysical fluids. Principal among these is the case with which synchronization is achieved in coupled Lorenz systems, its persistence and relative difficulty of desynchronization. This is perhaps significant in understanding the effectiveness of teleconnections, in which weak signals from distant features can apparently have significant and sometimes dramatic effects on other weather systems.

It should be mentioned here that the results presented in this paper were obtained for one-way coupled systems, i.e., the first system (4a)–(4c) affects the second on (4d)–(4f) but the second system has no influence on the first one. Usually in geophysical systems one has to consider mutual coupling, for example

$$\dot{X}_1 = -\sigma X_1 + \sigma Y_1 + e(X_2 - X_1), \tag{5a}$$
$$\dot{Y}_1 = -X_1 Z_1 + r_1 X_1 - Y_1 + e(Y_2 - Y_1), \tag{5b}$$
$$\dot{Z}_1 = X_1 Y_1 - b Z_1 + e(Z_2 - Z_1), \tag{5c}$$
$$\dot{X}_2 = -\sigma X_2 + \sigma Y_2 + d(X_1 - X_2), \tag{5d}$$
$$\dot{Y}_2 = -X_2 Z_2 + r_2 X_2 + d(Y_1 - Y_2), \tag{5e}$$
$$\dot{Z}_2 = X_2 Y_2 - b Z_2 + d(Z_1 - Z_2), \tag{5f}$$

where e is constant. As shown in [20], Eq. (4) can be considered as an approximation of Eq. (5) in the case where $e \ll d$, i.e., the influence of the first system on the second one is much larger than the vice versa influence. The dynamics of mutually coupled system was investigated in [11] and it shows very similar features.

The procedure of controlling chaos by chaos might be regarded as a possible mechanism for the so-called extended-range atmospheric predictability sometimes observed in geophysical systems. The results of this simple coupling show the great potential influence of the behaviour of one chaotic system over another. It is also clear that, in a stationary state, when the variables $\boldsymbol{x}$ and $\boldsymbol{y}$ are close together, the control signals $d(y - x)$ are small. This probably means that such a coupling might not allow for an easy experimental verification in real geophysical systems. It is, however, a mechanism which may account for the existence of many unexpected short and relatively predictable features in the situations when strongly chaotic behaviour might be expected. We hope that the results of this letter could be of direct benefit concerning the problem of "predicting predictability", which is currently a research topic of great interest. Finally, we remark that a similar chaos by chaos controlling mechanism, having a different type of coupling, was recently proposed to be responsible for stabilization of the Earth's obliquity by the Moon [19].

References

[1] H. Fujisaka and T. Yamada, Prog. Theoret. Phys. 69 (1983) 32.
[2] L. Pecora and T.L. Carroll, Phys. Rev. A 44 (1991) 2374.
[3] N.F. Rulkov, M.M. Sushchik and L.S. Tsimring, Phys. Rev. E 51 (1995) 980.
[4] K. Pyragas, Phys. Lett. A 181 (1993) 203.
[5] R. Brown, H.F. Rulkov and N.B. Tufillaro, Phys. Lett. A 196 (1994) 201.
[6] T. Kapitaniak, Phys. Rev. E 50 (1994) 1642.
[7] M. Sekieta and T. Kapitaniak, Bifurcation and Chaos 6, in press.
[8] Y. Liu and J.R. Rios Leite, Phys. Lett. A 191 (1994) 1647.
[9] Y. Liu, P.C. de Oliveira, M.B. Danailov and J.R. Rios Leite, Phys. Rev. A 50 (1994) 7843.
[10] Y. Liu and L.C. Barbosa, Phys. Lett. A 197 (1995) 13.
[11] J. Brindley, T. Kapitaniak and L. Kocarev, Geophys. Res. Lett. 22 (1995) 1257.
[12] J. Brindley and T. Kapitaniak, Enhanced predictability in chaotic geophysical systems, in: Proc. 3rd Tech. Conf. on Nonlinear Dynamics (Chaos) and Full Spectrum Processing, Mystic 95, American Institute of Physics, in press.
[13] E.H. Lorenz, J. Atmospheric Sci. 20 (1963) 130.
[14] T.N. Palmer, Bull. Amer. Meteorological Soc. 74 (1993) 49.
[15] R. Madden and P.R. Julian, J. Atmospheric Sci. 28 (1971) 702.
[16] K. Weickmann, J. Geophys. Res. 96 (1991) 3187.
[17] B.J. Hoskins and B. Karoly, J. Atmospheric Sci. 38 (1981) 1179.
[18] J.M. Wallace and D.S. Gutzler, Mon. Wea. Rev., 109 (1981) 785.
[19] J. Laskar, F. Joutel and P. Robutel, Nature 361 (1993) 615.
[20] L. Kocarev and T. Kapitaniak, J. Phys. A. 28 (1995) 249.