Loss of Chaos Synchronization through the Sequence of Bifurcations of Saddle Periodic Orbits

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In the work we investigate the bifurcational mechanism of the loss of stability of the synchronous chaotic regime in coupled identical systems. We show that loss of synchronization is a result of the sequence of soft bifurcations of saddle periodic orbits which induce the bubbling and riddling transitions in the system. A bifurcation of a saddle periodic orbit embedded in the chaotic attractor determines the bubbling transition. The phenomenon of riddled basins occurs through a bifurcation of a periodic orbit located outside the symmetric subspace. [S0031-9007(97)03692-2]

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Interactive chaotic systems are known to demonstrate the chaotic synchronization phenomenon [1–7]. In the case of identical systems a synchronous regime corresponds to a chaotic attractor that locates in the symmetric subspace $\mathbf{x}_1 = \mathbf{x}_2$ of the whole phase space of the system. When the system exits from the synchronization region, the chaotic attractor loses its stability in the normal to the subspace direction according to the determined scenario [8–11]. In this case bubbling and riddling transitions can be observed as intermediate stages. After bubbling transition the intermittent transient process can take place in the system. There are orbits repelled from the chaotic attractor and returned to the vicinity of the symmetric subspace. In this situation the noise of small intensity induces the so-called bubbling attractor [8]. After riddling transition in the basin of the symmetric chaotic attractor (including small neighborhood of an attractor) there appears a set of "holes" which belongs to the basin of other attractor [12–14].

These phenomena take place in different systems and are intensively investigated for the last time [8–19].

The loss of stability of the chaotic state in the normal direction is immediately connected with bifurcations of saddle periodic orbits embedded in the symmetric chaotic attractor. For instance, in the work [17] it was demonstrated that riddling transition in a symmetric system appears as a result of saddle-repeller subcritical bifurcation (the eigenvalue $+1$) of the saddle fixed point embedded in the chaotic attractor.

In this work we investigate the mechanism of the stability loss of the chaotic in-phase regime in the coupled logistic maps. We demonstrate that the loss of synchronization is a result of a sequence of soft bifurcations of the certain family of saddle periodic orbits. These bifurcations induce bubbling and riddling transitions in the system. A bifurcation of a saddle periodic orbit embedded in the chaotic attractor determines the bubbling transition. The phenomenon of riddled basins occurs through a bifurcation of a periodic orbit located outside the symmetric subspace.

Let us consider the system of two coupled logistic maps in the form

$$\begin{aligned} x_{n+1} &= \lambda - x_n^2 + \varepsilon(x_n^2 - y_n^2), \\ y_{n+1} &= \lambda - y_n^2 + \varepsilon(y_n^2 - x_n^2), \end{aligned} \tag{1}$$

where x_n, y_n are dynamical variables, λ is the controlling parameter of the single map, and ε is the coefficient of coupling.

With the increase of the parameter λ the system (1) is known to demonstrate the cascade of period-doubling bifurcations of symmetric ($x_n = y_n$) periodic orbits $2^N C^0$ ($N = 0, 1, 2, \ldots$) in the wide region of the coupling coefficient values [20]. The cascade of the bifurcations leads to creation of a chaotic attractor located in the symmetric subspace. With further increase of the parameter λ the band-merging bifurcations of the chaotic attractors $2^N A^0$ take place, and windows of stable periodic orbits of different periods exist. If we decrease the coupling coefficient, the synchronous chaotic oscillations are changed by nonsynchronous regimes [20–22]. Near the bifurcational point the intermittency of Yamada-Fujisaka takes place [23,24]. In the system (1) there are regions of the parameters' values where the multistability phenomenon is observed [21,25]. Formation of multistability occurs due to bifurcations of periodic orbits which take place when the parameters are changing. Every orbit can undergo several bifurcations. This leads to multiplication of families of the periodic orbits [26], many of which can coexist in the stable state under the certain parameters' values.

Let us consider the mechanism of destruction of synchronous chaotic motions in the coupled logistic maps from the point of view of bifurcations of the saddle periodic orbits embedded in the chaotic attractor.

In the system (1) at $\lambda = 1.56$ in the interval of values ε from 0.2043 to 0.7957 a one-band symmetric ($x_n = y_n$) chaotic attractor A^0 is observed. It is formed as a result of a cascade of the period-doubling bifurcations of the symmetric periodic orbits $2^N C^0$. In the mentioned interval of the parameters values these orbits are saddle ones. They

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are embedded in the attractor A^0 and determine its structure. When exiting from the region of synchronization saddle orbits of the family $2^N C^0$ lose their stability in the normal direction. With the decrease of the coupling coefficient the orbits C^0, $2C^0$, $4C^0$, $8C^0$, and $16C^0$, which are not changing their coordinates undergo bifurcations at values $\varepsilon = 0.2043$, 0.1659, 0.1614, 0.1628, and 0.1622, respectively.

The loss of stability of the chaotic synchronization regime begins with a bifurcation of the fixed point C^0. At $\varepsilon = 0.2043$ its minimal eigenvalue becomes equal to -1. There is the period-doubling bifurcation. As a result C^0 transforms to a repeller and in its neighborhood outside the symmetric subspace a saddle orbit of double period $2C^1$ softly appears [Fig. 1(a)]. It is symmetric to the transformation of coordinates $(x_n, y_n) \leftrightarrow (y_n, x_n)$. With the decrease of the coupling the points of the orbit continuously move away from the symmetric subspace. Now, in the small neighborhood of C^0 there are initial conditions starting from which trajectories leave the neighborhood of the symmetric subspace to the saddle orbit $2C^1$. In the neighborhood of every preimage of the point C^0 there are also regions of initial conditions of normal unstability. The saddle orbit $2C^1$ and its unstable manifolds bound the region near the symmetric subspace which the trajectories cannot leave. The phase point returns to the neighborhood of the symmetric chaotic set along the unstable manifolds and is attracted to it. The period-doubling bifurcation of the saddle point C^0 induces the bubbling transition in the system. After it, transient processes that have character of intermittency can be observed in the considered system. They are finished by the synchronous chaotic oscillations. The noise of small intensity induces bubbling attractor.

At lower parameter values there are period-doubling bifurcations of orbits of higher periods $2C^0$, $4C^0$, $8C^0$, and $16C^0$ embedded in the chaotic attractor. Their minimal eigenvalues become equal to -1. As a result they become repeller and saddle orbits of double periods softly appearing in its neighborhoods. Then points of the saddle orbits move away from the symmetric subspace. Its neighborhood becomes more riddled by holes in which phase point repels from it. The bubbling attractor becomes more developed. However, without fluctuations symmetric chaotic oscillations remain in the system after transient processes up to $\varepsilon \approx 0.154$.

We must underline that we have investigated bifurcations of orbits up to the period 16. But there are reasons to suppose that $2^N C^0$ family orbits of higher periods undergo the bifurcations in the considered region of the parameter values.

The process of further stability loss of the synchronous chaotic state A^0 is determined by bifurcations of saddle periodic orbits located outside the symmetric subspace but appeared as a result of bifurcations of periodic orbits embedded in the symmetric chaotic set A^0. At $\varepsilon = 0.1533$ the maximal eigenvalue of the saddle orbit $2C^1$ enters the unit circle through $+1$. It becomes stable and a pair of saddle orbits of the same period $2C_1^s$ and $2C_2^s$ softly appears in its neighborhood. They are symmetric to each other according to transformation of coordinates $(x_n, y_n) \leftrightarrow (y_n, x_n)$. At reverse parameter changing this bifurcation corresponds to the subcritical pitchfork bifurcation. The bifurcation of the saddle orbit $2C^1$ which has appeared from the embedded into the chaotic attractor fixed point C^0 induces the riddling transition in the system (1). In Fig. 2 there are basins of the symmetric chaotic set A^0 (white) and of the stable periodic orbit $2C^1$ (black). From the figure one can

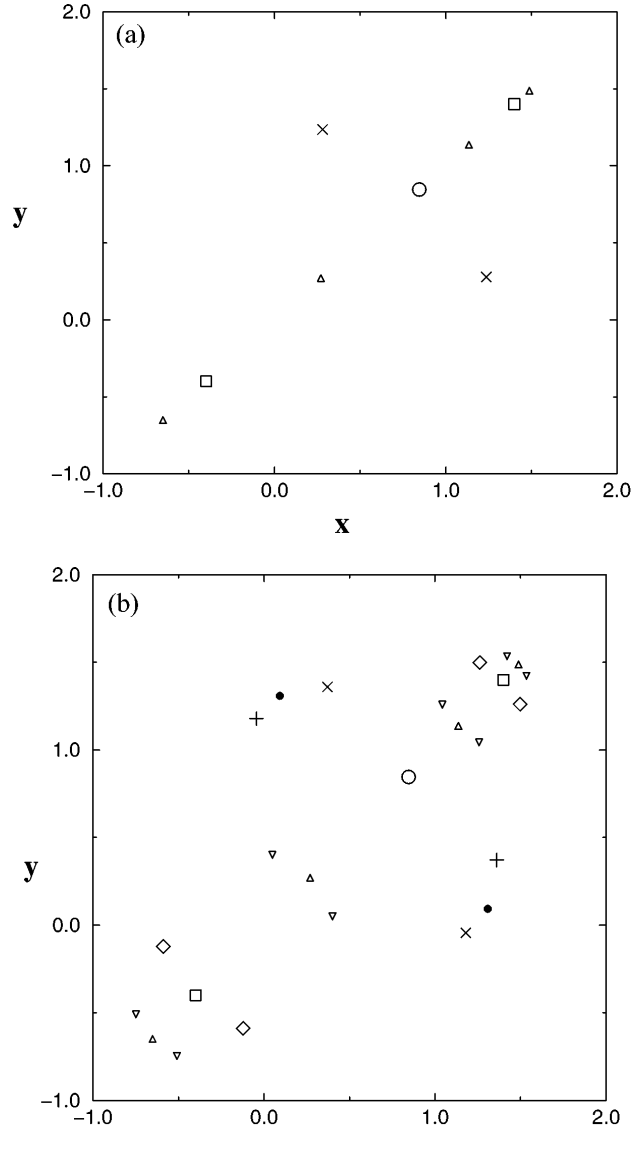

FIG. 1. (a) The repeller C^0 (○), the saddles $2C^0$ (□), $4C^0$ (△) embedded in the chaotic set A^0 and the saddle $2C^1$ (×) the bifurcation of which induces the bubbling transition ($\lambda = 1.56, \varepsilon = 0.17$). (b) The repellers C^0 (○), $2C^0$ (□), $4C^0$ (△) and the saddles $4C^2$ (◇), $8C^4$ (▽) appeared in the result of bifurcations of $2C^0$ and $4C^0$, respectively. The saddles $2C_1^s$ (×) and $2C_2^s$ (+) the birth bifurcation of which transforms $2C^1$ to the stable node (●) and induces the riddling transition ($\lambda = 1.56, \varepsilon = 0.143$).

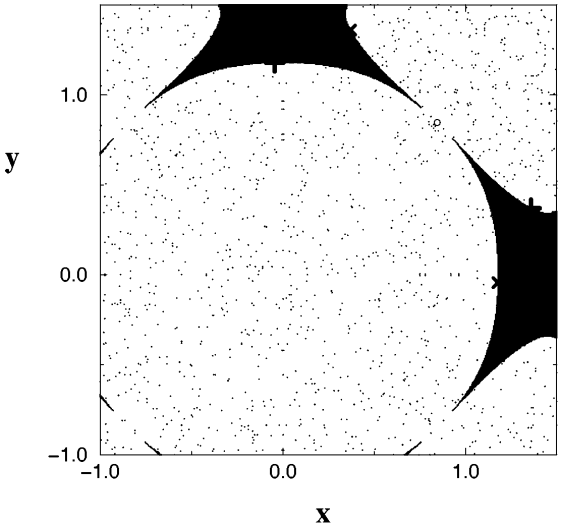

FIG. 2. Basins of attracting of the chaotic set A^0 (white) and of the stable periodic orbit $2C^1$ (black) obtained at $\lambda = 1.56, \varepsilon = 0.143$ in the region $-1 \leq y \leq 1.5$; $-1 \leq x \leq 1.5$ with step 0.005 and the maximal number of iterations $n = 10^6$. The repeller C^0, the stable node $2C^1$, and the saddles $2C_1^s$, $2C_2^s$ are marked by "(○)", "(●)", "×", and "+", respectively.

see that the basin of the chaotic set A^0 becomes riddled by holes that relate to the basin of the attractor $2C^1$. As a result of bifurcation of the saddle orbit $2C^1$ in the phase space of the system there appears a region which narrows to the symmetric subspace and leans on the repeller C^0. In this region phase trajectories leave the chaotic attractor and are attracted to $2C^1$. Near every preimage of the fixed point C^0 there are also such regions. This leads to riddling of the basin of the chaotic set A^0. The case of creation of the regions of capture of phase trajectories by another attractor is similar to that described in the work [17]. However, in our case we have another bifurcational mechanism.

In the system (1) the riddling transition is determined by two consecutive bifurcations that are schematically described in Fig. 3. The embedded in the chaotic attractor saddle fixed point C^0 undergoes the period-doubling bifurcation [Fig. 3(a)]. In its neighborhood the saddle orbit $2C^1$ appears [Fig. 3(b)]. Then $2C^1$ undergoes a bifurcation in the result of which a pair of the saddle orbits $(2C_1^s, 2C_2^s)$ appears in its neighborhood and $2C^1$ becomes stable [Fig. 3(c)]. In this case the "tongue" of capture of phase trajectories by $2C^1$ is formed. Its bounds are the stable manifolds of the saddle orbits $2C_1^s$ and $2C_2^s$. In the moment of the bifurcation the tongue is infinitely narrow. With moving away from the bifurcational value $2C_1^s$ and $2C_2^s$ diverge and the tongue continuously expands.

At $\varepsilon = 0.1274$ and 0.1238 the saddle orbits $8C^4$ and $4C^2$ undergo the same bifurcations. This leads to further riddling of the basin of A^0. In its neighborhood there appears an additional set of points that belong to the basin of $4C^2$.

With further decrease of the coupling coefficient phase trajectories depart to attractors relating to nonsynchronous regimes practically from any initial conditions. The set A^0 transforms to a chaotic saddle.

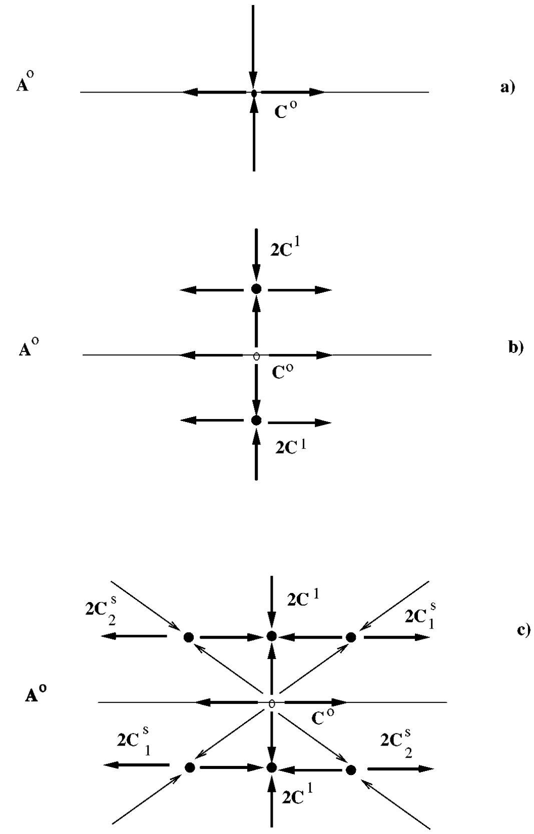

FIG. 3. Consecutive bifurcations that induce the bubbling (a), (b) and the riddling (b), (c) transitions.

In the other considered region of the values of the coupling coefficient, when ε is more than 0.7957 we observed a similar scenario of the loss of stability of A^0. However, there are differences in bifurcations of some saddle orbits, but they also lead to bubbling and riddling transitions. At $\varepsilon = 0.7957$ the saddle point C^0 undergoes the supercritical pitchfork bifurcation but not the period-doubling bifurcation. Its minimal eigenvalue becomes equal to $+1$. The saddle C^0 transforms to repeller. In its neighborhood outside the symmetric subspace a pair of saddle fixed points C^1 and C^2 appear. This bifurcation induces the bubbling transition in the system. At $\varepsilon = 0.8467$ the maximal eigenvalues of the saddle points C^1 and C^2 enter the unit circle through -1. They become stable and in their neighborhoods saddle orbits of the double period appear. At reverse parameter changing this case corresponds to the subcritical bifurcation of period doubling. The bifurcations of the saddle points C^1 and C^2 induce the riddling transition in the system (1). Saddle orbits $2C^0$, $4C^0$, $8C^0$, and $16C^0$ undergo the same bifurcations as in the region of small coupling.

In this work we carried out investigations of stability loss mechanism of synchronous chaotic state in the coupled logistic maps from the point of view of bifurcations of saddle periodic orbits. We showed that the exit from the region of synchronization is followed by the sequence of soft bifurcations of the family of saddle orbits $2^N C^0$ which form the structure of the attractor. The loss of stability of the symmetric chaotic set in the normal direction begins with bifurcation of the saddle point C^0 that induces the bubbling transition in the system. The saddle periodic orbit and its unstable manifolds which appeared from C^0 bound the region near the symmetric subspace, from which a trajectory cannot leave from. Soft bifurcation of this fixed point located outside the symmetric subspace induces the riddling transition in the system. Bifurcations of saddle orbits of higher periods intensify the bubbling and riddling effects. The basin of the symmetric attractor can be riddled by holes a part of which belongs to the basin of one regular attractor and a part belongs to the basin of the other one. With further change of the control parameter value away from the region of synchronization the chaotic attractor continuously "loses" its basin and transforms to chaotic saddle. We suppose that the bifurcational scenario described in the work is rather common for identical coupled systems with period doubling.

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