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# Practical Stability of Chaotic Attractors

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**Abstract**—In this paper we introduce the concept of practical stability and practical stability in finite time for chaotic attractors. The connection between practical and asymptotic stability is discussed.


## 1. INTRODUCTION

One of the fundamental problems in practical applications of chaotic dynamics is the problem of a stability of a chaotic attractors $A$, e.g. chaos synchronization [1–10]. The basin of attraction $\beta(A)$ is the set of points whose $\omega$-limit set is contained in $A$. In Milnor's definition of an attractor [11], the basin of attraction need not include the whole neighbourhood of the attractor, i.e. we say that $A$ is a Milnor attractor if $\beta(A)$ has positive Lebesgue measure. However, riddled basins, which have recently been found in practical physical systems [12–16] have positive Lebesgue measure yet do not contain any neighbourhood of the attractor. If the basin of attraction contains the neighbourhood of $A$, then the attractor is asymptotically stable.

Most of the chaotic attractors which can be met in practical engineering systems are quasi-attractors, i.e. the limiting sets enclose the periodic orbits of different topological types, structurally unstable homoclinic trajectories, etc. Practical systems are mainly quasi-hyperbolic [17], i.e. many different types of attractors co-exist in the phase space.

Existing definitions of the stability of chaotic attractors cannot always give sufficient practical information about the behaviour of the real engineering system which is under the influence of both permanently acting and short-time impulse-like perturbations. The main problems in stability analysis of chaotic engineering systems are as following.

- The basin of attraction of the asymptotically stable chaotic attractor can be so small that perturbations can take a trajectory out of it to the basin of another attractor.
- The system operates in finite time $T$ during which the system cannot reach the attractor, but for $t < T$ the system evolves in a limited part of the phase space which does not necessarily include the final attractor of the system [18].

In this paper we introduce the concept of practical stability and practical stability in finite time for chaotic attractors.

The paper is organized as follows. In Section 2 we introduce definitions of practical stability and practical stability in finite time of chaotic attractors. Examples and the similarities and differences between asymptotic and practical stability of chaotic attractors are discussed in Section 3. In this section we also describe the controlling procedure which allows some asymptotically unstable attractors to be practically stable. Finally, we summarize our results in Section 4.

## 2. DEFINITIONS

Consider a dynamical system given by

$$\frac{dx}{dt} = f(x,t) \tag{1}$$

which for initial conditions $x(t_0) = x_0 \in \omega$, where $\omega$ is an open set, has asymptotically stable attractor $A \in \mathfrak{R}^n$.

**Definition 1.**

(1) Let the system (1) be under the influence of permanently acting perturbations $p(x,t)$ so the perturbed system is in the form

$$\frac{dx}{dt} = f(x,t) + p(x,t). \tag{2}$$

(2) Let the perturbation function $p(x,t)$ fulfil the condition

$$\|p(x,t)\| \le \delta,$$

where $\delta > 0$.

(3) Let $\Omega$ be a closed, bounded set such that $A \in \Omega$ and $\omega \in \Omega$.

If, for all initial conditions $x(t_0) = x_0 \in \omega$, all functions $p(x,t)$ and all $t \ge t_0$, $x(t) \in \Omega$, then the attractor $A$ is practically stable (in relation to sets $\omega$, $\Omega$ and perturbations $p(x,t)$).

In this definition, the function $p(x,t)$ describes all continuously acting perturbations. The set $\omega$ defines the limits of both uncertainties in initial conditions and short-time perturbations. Perturbed trajectories $x(t)$ of the system (2) evolve in the region of the phase space given by the set $\Omega$, which is usually larger than the attractor $A$ of the unperturbed system (1).

If the attractor $A$ is a fixed point [19], then in the absence of permanently acting perturbations ($p(x,t) = 0$), definition 1 is equivalent to the definition of the stability in the sense of Lagrange in relation to the set $\Omega$. In every case the practical stability is independent of the stability in the sense of Lyapunov.

Many engineering systems operate in a finite time and for stability investigations of such systems we can introduce a weaker definition.

**Definition 2.**

Let conditions (1)–(3) of definition 1 be fulfilled.

If, for all initial conditions $x(t_0) = x_0 \in \omega$, all functions $p(x,t)$ and all $t_0 \le t < T$, $x(t) \in \Omega$, then the attractor $A$ is practically stable in finite time $T$ (in relation to sets $\omega$, $\Omega$ and perturbations $p(x,t)$).

## 3. EXAMPLES

### 3.1 *Spiral and double-scroll attractors of Chua's circuit*

As an example let us consider the dynamics of Chua's circuit [20, 21]. Chua's circuit is an RLC circuit with four linear elements (two capacitors, one resistor, and one inductor) and a nonlinear diode, which can be modelled by a system of three differential equations. The equations for this circuit are

$$\frac{dx}{dt} = \alpha(y - x - g(x))$$
$$\frac{dy}{dt} = x - y + x \qquad (3)$$
$$\frac{dz}{dt} = \beta y$$

where the piecewise linear function

$$g(x) = bx + 0.5(a - b)(|x + 1| - |x - 1|)$$

describes three different voltage–current regimes of the nonlinear diode. Parameters $\alpha$, $\beta$, $a$ and $b$ are constant.

Two of the best-known attractors of system (3), the spiral and double-scroll attractors, are shown in Fig. 1. They can be observed for $\beta = 14.87$, $a = -1.42$, $b = -0.68$, $\alpha = 7.7$ (spiral) and $\alpha = 10$ (double-scroll). In the case of the spiral attractor there are two symmetrical co-existing attractors $A_1$ and $A_2$. All attractors in Fig. 1 are shown together with their basins of attraction $\beta(A_1)$, $\beta(A_2)$ and $\beta(B)$, respectively, and one can easily see that they are asymptotically stable. In Fig. 1 the two-dimensional $x$–$z$ cross-sections of the three-dimensional basins of attraction defined by $y = 0$ are shown. Attractors $A^+$, $A^-$ and $B$ are projected into these cross-sections.

If we define sets $\omega$ and $\Omega$ as in Fig. 2(a), and allow perturbations $p(x,t)$ to evolve only in $\Omega$, attractor $A_1$ is practically stable in relation to sets $\omega$, $\Omega$ and perturbations $p(x,t)$. If these sets $\omega$ and $\Omega$ are too small from a practical point of view, and we have to consider sets like these in Fig. 2(b), the attractor is no longer practically stable.

In the case of $\alpha = 10$, the double-scroll attractor is the only chaotic attractor of the system (3), and both spiral attractors are unstable (they do not exist). Now we describe the controlling procedure based on the Ott–Grebogi–Yorke (OGY) method [22] in the version given in Ref. [23]. Let us consider the sets $\omega$ and $\Omega$ shown in Fig. 3 and perturbations $p(x,t)$ evolving only in $\Omega$. If we want the trajectory to evolve, say, on the $A_1'$ part of the attractor, we can define the dangerous zones $\mathcal{D} \in B$ and $\mathcal{D} \in \Omega$ when the system visits before executing the undesirable evolution on $A_2'$. Our goal is to diverge the trajectory entering the $\mathcal{D}$ region out of it and back to the set $\Omega$. To achieve this goal we assume that one of the system parameters, let us say $\alpha$, can be adjusted finely around a nominal value $\alpha_0$, i.e. $\alpha \in [\alpha_0 + \Delta\alpha, \alpha_0 - \Delta\alpha]$, where $\Delta\alpha/\alpha_0 \ll 1$.

To control the system one can build the return map $X_{n+1} = f(X_n, \alpha)$ by plotting successive extrema of the observed map. The dangerous zone $\mathcal{D}$ is determined by observing the iterates of the system as it approaches the undesirable part of the attractor $B$. The extent of $\mathcal{D}$ is determined by the distribution of points in that zone. We then pick $\mathcal{D}_m$, the dangerous zone to implement control on, where $\mathcal{D}_m$ is the interval in the return map composed of the $m$th pre-iterates which occur before the trajectory leaving $\Omega$.

Following the OGY method [22], we change slightly the control parameter $\alpha$ so that the

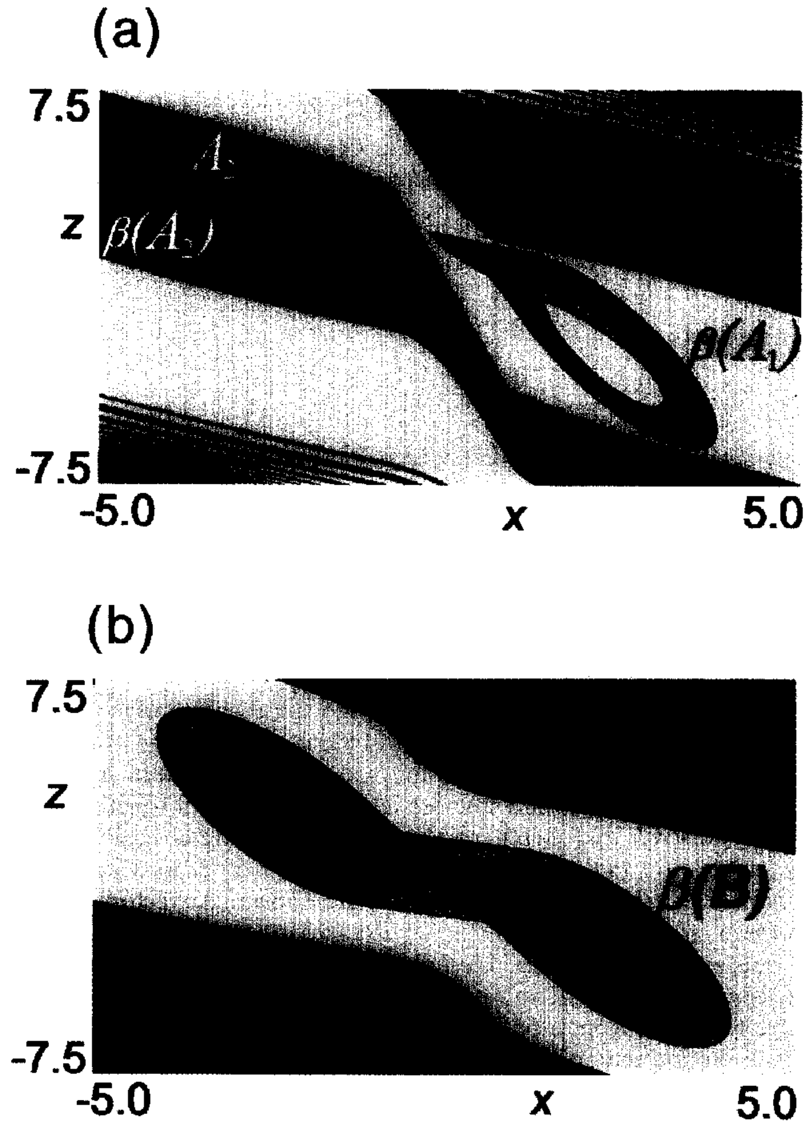


Fig. 1. Attractors of Chua's circuit: $\beta = 14.87$, $a = -1.42$, $b = -0.68$; (a) $\alpha = 7.7$ (spiral attractor), (b) $\alpha = 10$ (double-scroll attractor). The two-dimensional $x$–$z$ cross-sections of the three-dimensional basins of attraction defined by $y = 0$ are shown. Attractors $A^{1}$, $A$ and $B$ are projected into these cross-sections.

attractor moved and observed the resulting change in location of the next iterate which corresponds to the $(m-1)$th pre-iterate before departure from $\Omega$, which we call $\mathcal{D}_{m-1}$. From this observation one can calculate the shift of the dangerous zone $\mathcal{D}_{m-1}$ per unit change of the control parameter as [23]

$$g = \frac{\partial f(X_n, \alpha)}{\partial \alpha}.$$

The control is implemented when the system enters the dangerous zone $\mathcal{D}_m$. As the location of the region $\mathcal{D}_{m-1}$ is known from previous observations, it is possible to calculate $d_{m-1}$ defined as the minimum distance by which the attractor has move such that the next

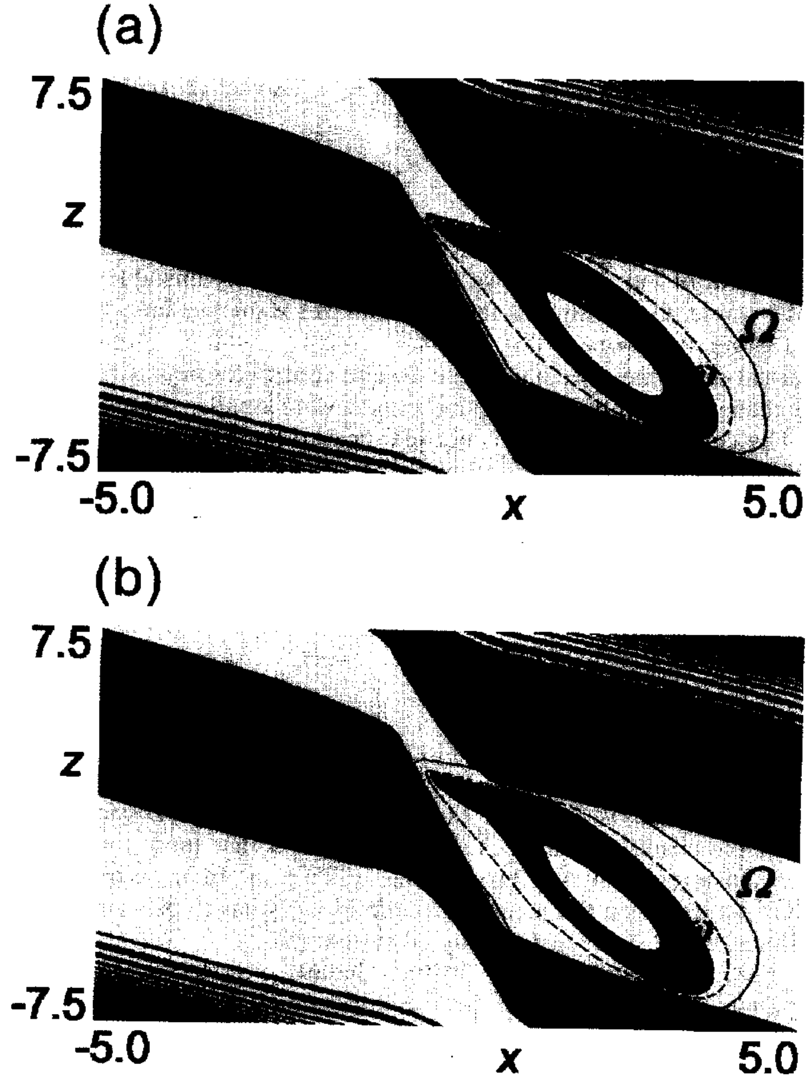

Fig. 2. Examples of practically stable (a) and practically unstable (b) attractors.

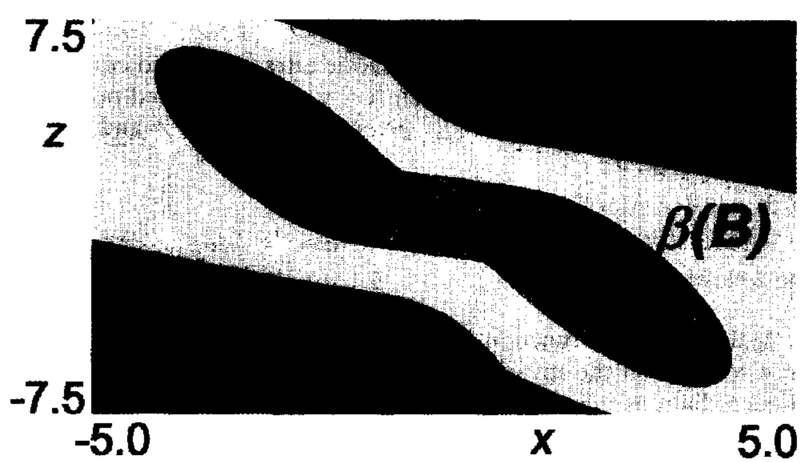

Fig. 3. Idea of controlling method.

iterate of the return map falls out of the extent of $\mathcal{D}_{m-1}$ into set $\Omega$. With this distance we can calculate the corresponding $\alpha$ parameter change

$$\partial\alpha = \frac{d_{m-1}}{g}. \tag{4}$$

The $\alpha$ parameter change given by eqn (4) is an occasional feedback control which, applied to the system of Fig. 2(b), allows us to make originally unstable spiral attractors $A_1$ or $A_2$ practically stable. Generally this method stabilizes a smaller unstable chaotic attractor embedded in a larger stable one.

In the numerical experiment we have been able to control a spiral attractor like the one in Fig. 1(a). The changes in controlling parameters $\partial\alpha$ were smaller than 4% of their nominal value $\alpha = 10$. The controlled attractor is not identical to the spiral attractor $A_1$ of Fig. 1(a), but with definition 1 the evolution shown in this figure can be considered as the evolution on a practically stable attractor $A_1$.

### 3.2 *Stability of synchronized chaotic attractors*

The application of the above definitions will be illustrated for the example of unidirectionally coupled systems

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \tag{5a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y) + \mathrm{d}(x - y) \tag{5b}$$

where $x,y \in \mathfrak{R}^n$ and $d \in \mathfrak{R}^n$ is constant. For $d = 0$ both $x$ and $y$ subsystems evolve on the asymptotically stable chaotic attractor $A$. It is well-known that there exists a value of $d$ for which $x$ and $y$ subsystems can synchronize, i.e. $x = y$ [10]. In the synchronized state the chaotic attractor $A$ located on the invariant manifold $x = y$ has to be asymptotically stable in the $2n$-dimensional phase space of the coupled system (5).

In this section we give conditions under which the attractor $A$ can be practically stable.

Introducing a new variable $e = x - y$, we can rewrite eqns (5a) and (5b) as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \tag{6a}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = f(x) - f(y) - \mathrm{d}e \tag{6b}$$

where eqn (6a) describes the evolution on the chaotic attractor $A$ and eqn (6b) the evolution transverse to it. In the synchronized state, when $x = y$, $e = 0$ is a fixed point of eqn (6b).

Suppose there exists a function $V(e,t) \in \mathscr{C}^1$ given for all $e,x,y \in A$ and $t \geq 0$ such that:

(1) $V(e,t) > 0$ for $e \neq 0$;
(2) $\partial V(e,t)/\mathrm{d}t + [\partial V(e,t)/\partial e][\mathrm{d}e/\mathrm{d}t] \leq 0$ for $e \in \mathfrak{R}^n - \omega$;
(3) $V(e_1,t_1) < V(e_2,t_2)$ for $e_1 \in \omega$, $e_2 \in \mathfrak{R}^n - \Omega$ and $t_1 < t_2$.

We have the following result:

**Theorem 1.** If there exists a function $V(e,t)$ which fulfils conditions (1)–(3) for all $x \in A$, then attractor $A$ located on the invariant manifold $x = y$ is practically stable (in relation to sets $\omega$, $\Omega$ and perturbations $p(x,t)$.

*Proof*. Consider the solution $e(t)$ for initial conditions $x(t_0) \in \omega$. As set $\omega$ is open and $\omega \subset \Omega$, there exists $t_1 > t_0$ such that for all $x$, $y \in A$ and $e(t_1) \in \omega$. If for $t > t_1$ solution $e(t)$ for all $x \in A$ stays in $\Omega$, then the theorem is true. Suppose that $x(t)$ leaves set $\Omega$ and, for $t_2 > t_1$, $e(t_2) \in \mathfrak{R}^n - \Omega$. From condition (3) one can write

$$V[e(t_1),t_1] < V[e(t_2),t_2].$$

This means that the function $V(e,t)$ grows along $e(t)$, but this is impossible as, based on conditions (1) and (2), $V(e,t)$ is a non-growing function. This proves the theorem. □

For the case of practical stability in finite time, suppose there exists a function $V(e,t) \in \mathscr{C}^1$ given for all $e,x,y \in A$ and $t \geq 0$ such that:

(1) $V(e,t) \geq 0$ for all $e$ and $t \geq 0$;
(2) $\partial V(e,t)/\mathrm{d}t + [\partial V(e,t)/\partial e][\mathrm{d}e/\mathrm{d}t] \leq 0$ for $e \in \mathfrak{R}^n - \omega$ and $0 \leq t_1 \leq t \leq t_1 + T$;
(3) $V(e_1,t_1) < V(e_2,t_2)$ for $e_1 \in \omega$, $e_2 \in \mathfrak{R}^n - \Omega$ and $t_1 < t_2 \leq t_1 + T$.
We have the following result:

**Theorem 2.** If there exists a function $V(e,t)$ which fulfils conditions (1)–(3) for all $x \in A$, then attractor $A$ located on the invariant manifold $x = y$ is practically stable in finite time $T$ (in relation to sets $\omega$, $\Omega$ and perturbations $p(x,t)$).

*Proof*. The proof is very similar to the proof of Theorem 1, so we omit it.

## 4. CONCLUSIONS

We hope that the concept of practical stability and practical stability in finite time can be very useful in the study of chaotic systems. Particularly, it can be useful in the study of chaos synchronization problems were the property of the practical stability of the synchronized chaotic state seems to be essential for any practical application of chaos synchronization, for example secure communication.

It was also shown that the appropriate control of asymptotically unstable systems can make them practically stable.

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