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Preserving transient chaos

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Abstract

We present a controlling method which allows either the preservation of transient chaos or a significant decrease of its lifetime. For the description of transient chaotic evolution we introduce practical Lyapunov exponents. Success of the method requires some knowledge, based on observation, of the character of the chaotic repeller which drives the transient chaos, but its application is then straightforward. © 1998 Elsevier Science B.V.

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1. Introduction

Most of the chaotic attractors which are met with in practical systems are quasi-attractors, i.e., the limiting sets of enclosing periodic orbits of different topological types, structurally unstable homoclinic trajectories, etc. [1]. Practical systems are mainly quasi-hyperbolic [2], i.e., many different types of attractors co-exist in the phase space. In such systems we often observe the phenomenon of transient chaos, where for almost all initial conditions within some practically important range, the system trajectory evolves on a strange chaotic repeller for significantly long period of time, τ say, and afterwards, for $t > \tau$, converges to the regular attractor [3,4]. The value of τ will of course vary from trajectory to trajectory, and may be very sensitive to the initial conditions, but we will in the rest of this Letter assume τ to be some representative average.

From a practical point of view, there is often not much difference between transient and permanent chaotic behavior as, for example, (i) the working time (T) of engineering devices can be shorter than τ and a system cannot reach its final attractor; (ii) the lifetime of transient chaos τ is longer than any reasonable observation period of time (T) in biological or geophysical systems.

On the other hand, practical systems are always under the influence of both permanently acting and short time impulse-like perturbations. These perturbations can affect the system evolution, causing a switch to another co-existing attractor, or can significantly increase or decrease the transient lifetime τ, in some cases leading to a catastrophic failure in a practical sense.

The above reasons make the problem of controlling transient chaos important not only theoretically but also from the point of view of possible practical applications. In a pioneering work [5] Tel showed that transient chaos can be controlled by the Ott–Grebogi–

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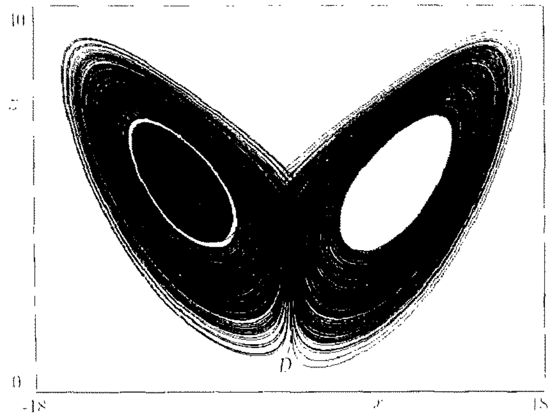

Fig. 1. Transient chaotic evolution of system (2): $\sigma = 10$, $b = 8/3$, $r = 23.2$, $x(0) = 1$, $y(0) = 0$, $z(0) = 45$.

Yorke method [6] and replaced by an appropriate periodic behavior. In this Letter, we take a different approach and present a controlling method which can also allow preservation of transient chaos.

As an example, we consider the dynamics of a particular but representative model of a continuous system,

$$\dot{X} = f(X, r), \tag{1}$$

where the system parameter $r \in \mathbb{R}$ and $X \in \mathbb{R}^n$, namely the Lorenz system

$$\frac{dx}{dt} = -\sigma(x - y),$$

$$\frac{dy}{dt} = -xz + rx - y,$$

$$\frac{dz}{dt} = xy - bz, \tag{2}$$

where σ, b, $r \in \mathbb{R}$ are constant. It is well known that the system (2) exhibits transient chaotic behavior for $r \in (13.96, 24.74)$. In this case a system trajectory evolves for a significantly long period of time on a strange chaotic repeller (with a shape similar to the well-known Lorenz attractor) before converging to one of the fixed points $C_{1,2} = (\pm[b(r-1)]^{1/2}, \pm[b(r-1)]^{1/2}, r-1)$. Such a trajectory is shown in Fig. 1, where the trajectory ultimately converges to $C_2 = (-[b(r-1)]^{1/2}, -[b(r-1)]^{1/2}, r-1)$.

2. Controlling procedure

In this section, we present the controlling procedure which can allow us either to preserve the transient chaotic behavior or, alternatively, to significantly reduce the lifetime of transient evolution. To achieve this goal we assume that one of the system parameters, let us say r, can be adjusted finely around a nominal value r_0, i.e., $r \in [r_0 + \Delta r, r_0 - \Delta r]$, where $\Delta r / r_0 \ll 1$.

Observation of the system behavior allows the determination of a phase space region which a trajectory $\Gamma(t)$ must enter shortly before converging to the fixed point. This region will be called a dangerous zone and will be indicated by $\mathcal{D}$. In the neighborhood of $\mathcal{D}$ one can identify a number, K say, of trajectories $\gamma_k(t)$ going out of this neighborhood and evolving further on the chaotic repeller. We can further identify a set $\mathcal{S}$ of points γ_k^* on these trajectories which we called a safe set. A schematic of zone $\mathcal{D}$ and set $\mathcal{S}$ estimated for the Lorenz system (1), is shown in Fig. 2.

In the Lorenz system the identification of the dangerous zone $\mathcal{D}$ is straightforward as it is well known [4] that the occurrence of the transient chaos is connected with the breakdown of the homoclinic orbit of unstable fixed point (0, 0, 0), and zone $\mathcal{D}$ therefore lies in the neighborhood of the unstable manifold of the fixed point (0, 0, 0). Alternatively $\mathcal{D}$ can be identified in the neighborhoods of stable fixed points $C_{1,2}$. Based on the same arguments, the dangerous zone can be identified in other systems like Duffing's, Chua's, etc. For example in Ref. [7], we identified the dangerous zone on a double-scroll attractor and applied control which forced the system to evolve on one scroll only.

$\mathcal{D}$ can be estimated also when the equations of motion are unknown and our knowledge of the system is based on a scalar time series. In this case, one can construct a return map $x_{n+1} = f(x_n)$ and identify pre-images of points going straight to the fixed points. There, pre-images define the dangerous zone (see, e.g. Ref. [8]).

Our controlling method which allows the preservation of transient chaos consists of three steps,

(i) estimation of the dangerous zone $\mathcal{D}$;

(ii) identification of trajectories $\gamma_k(t)$ in the neighborhood of $\mathcal{D}$ which allow further evolution on the chaotic repeller and creation of the safe set $\mathcal{S}$ of points γ_k^* in the ϵ-neighborhood of $\mathcal{D}$;

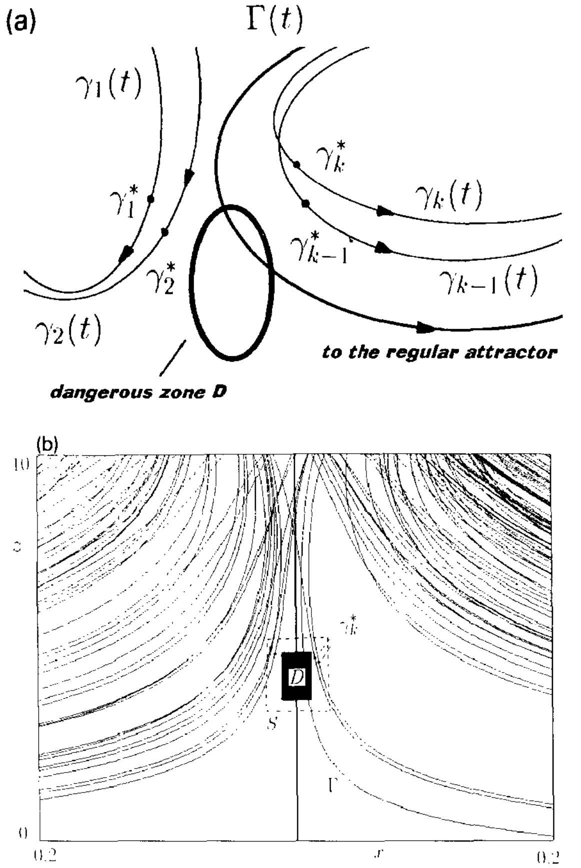

Fig. 2. (a) Idea of the controlling method, and (b) the dangerous zone $\mathcal{D}$ and safe set $\mathcal{S}$ for the system (2).

(iii) introduce a small temporal change of one of the system parameters to allow the system to switch from actual trajectory $\Gamma(t) \in \mathcal{D}$ to one of the safe $\gamma_k(t)$ trajectories.

To control a trajectory $\Gamma(t)$ which enters the dangerous zone $\mathcal{D}$ we use a simple feedback procedure. Suppose that the trajectory of the N-dimensional map $x_{n+1} = M(x_n, r)$ (the numerical solution of Eqs. (1) can be considered as an example of such a map), entering the set $\mathcal{D}$ falls into a ϵ-neighborhood of a point γ_k^* in the safe set $\mathcal{S}$, i.e., $|x_n - y_n| \leqslant \epsilon$, where y_n, representing the point γ_k^*, and its future iterates $y_{n+1}, y_{n+2}, \ldots$ constitute a safe trajectory going away from the dangerous set $\mathcal{D}$. In the neighborhood of x_n we can consider the following linearized dynamics

$$\Delta x_{n+1} = DM(x_n, r)\Delta x_n + \frac{\partial M}{\partial r}\Delta r_n, \tag{3}$$

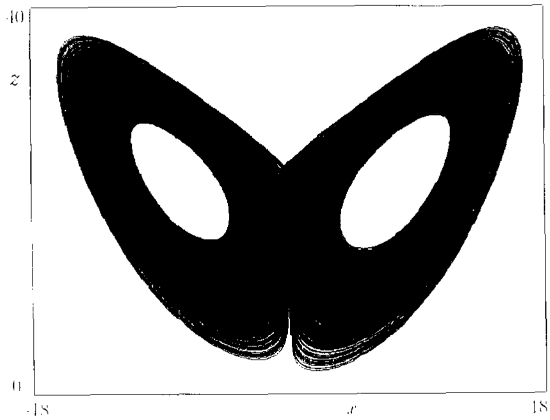

Fig. 3. Controlled trajectory of the system (2); $x(0) = 1$, $y(0) = 0$, $z(0) = 45$.

where $\Delta x_n = x_n - y_n$, $\Delta r_n = r_n - r_0$, and the Jacobian $DM(x_n, r)$, and the vector $\partial M/\partial r$ are calculated at $x_n = y_n$ and $r_n = r_0$ respectively. If we choose a unit vector u in the phase space and let u be orthogonal to $\Delta x_{n+1} = 0$, we obtain the r-parameter perturbation necessary to achieve control,

$$\Delta r_n = -\frac{uDM(x_n, r)\Delta x_n}{u\partial M/\partial r}. \tag{4}$$

The unit vector can be chosen arbitrarily provided that (i) it is not orthogonal to x_{n+1}, and (ii) the denominator in Eq. (4) is not zero.

We now use this approach to control the system (2). The set $\mathcal{D} = \{(x, y, z); -0.02 < x, y < 0.02, 3.5 < z < 4.5\}$ was taken as in Fig. 2b (without control, after the transient chaotic evolution of an average lifetime, here taken to be $\tau = 690$, trajectories converge to one of the fixed points). We assume that the accessible parameter r can be slightly perturbed around its nominal value $r_0 = 23.2$, and we take the maximum allowed parameter perturbation $\Delta r_{\max}$ to be 10^{-1}. We them create a safe set S, which consists of 200 points $\gamma_k^* \in \{(x, y, z) : -0.04 < x, y < 0.04, 3.49 < z < 4.51\} - \mathcal{D}$, and in the controlling procedure take $\epsilon = 10^{-2}$. With such a control we insure preservation of transient chaos. An example of controlled evolution is shown in Fig. 3.

A practical consideration is that the number of points γ_k^* in the safe set $\mathcal{S}$ should be large enough to avoid a situation in which the trajectory entering dangerous zone $\mathcal{D}$ will be switched to the same γ_k^* point all the time. In this case transient chaotic behavior can be replaced by long period periodic evolution. With a great number of γ_k^* points, even if a periodic trajectory is created, its period T_p will be long enough that from the point of view of practical applications it can be considered as chaotic. In this case (see below in Section 3) the topological and dynamical properties of chaos are preserved for all $t < T$, where $T (< T_p)$ is the time limit of operations or observations.

It should be noted here than the described method can be applied also to control transient chaos in the way to allow the quickest possible convergence to the fixed points. In this case our procedure can be "inverted" so that the dangerous zone $\mathcal{D}$ becomes a desired zone and all trajectories entering its neighborhood will be forced into it by the temporal change of system parameter. Controlling the transient evolution of the system (2) with the conditions given in the previous paragraph we managed to reduce the average lifetime of transient chaos to $\tau = 130$.

An alternative controlling method for preserving transient chaos was proposed by Lai et al. [9,10]. In their approach the OGY chaos controlling method was applied to stabilize a selected nonattracting chaotic trajectory, say $\bar{x}(t)$. The system trajectory $x(t)$ is forced to stay in the neighborhood of this selected trajectory by small temporal perturbations applied to the system whenever the distance between $\bar{x}(t)$ and $x(t)$ is larger than an assumed value ϵ. In our method we create an artificial chaotic trajectory which consists of parts of nonattracting chaotic trajectories imbedded in the chaotic repeller; the control is applied when the system trajectory enters the dangerous zone $\mathcal{D}$.

3. Practical Lyapunov exponents

The evolution of the system (1) on its attractor can be described by the spectrum of the Lyapunov exponents [4,7]. The Lyapunov exponents are given by the limit

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \|Y(t)\|, \qquad (5)$$

where $Y(t)$ is the solution of the linearized equation $\dot{Y} = \partial f/\partial x Y$ and $\|.\|$ is a norm in $\mathbb{R}^n$.

Lyapunov exponents given by the limit for $t \to \infty$ cannot be used for a description of the system evolution during the transient chaos. For this description we propose the average value of the transient Lyapunov exponents is given by

$$\langle \lambda(T) \rangle = \frac{1}{T - t_0} \int_{t_0}^{T} \lambda(t)\, dt, \qquad (6)$$

where

$$\lambda(t) = \frac{1}{t} \ln \|Y(t)\|. \qquad (7)$$

and $T < \tau$, i.e. practical Lyapunov exponents are calculated for as long as the trajectory evolves on the chaotic repeller.

Exponents given by the formula (6) describe the dependence on the initial conditions of trajectories evolving on the strange chaotic repeller (temporal divergence or convergence of nearby trajectories during transient chaotic evolution). We propose to call them practical Lyapunov exponents.

For the system (2) practical Lyapunov exponents for transient chaotic behavior ($T = 600$) are 0.72 ± 0.01, 0, -14.27 ± 0.01. For the controlled evolution of Fig. 3 we obtain the values 0.73 ± 0.01, 0, -14.25 ± 0.01 for $T = 6000$ and observe that these values do not change with the increase of T. In other words the chaotic behavior is maintained in a sort of "steady state" indefinitely.

Note that practical Lyapunov exponents are the same idea as truncated or local Lyapunov exponents [11,12]. In our case the time T for which they are estimated is determined by practical reasons rather than lack of data.

4. Conclusions

The method presented allows control of transient chaotic systems when our goal is to

(i) preserve transient chaos (not allowing the system to converge to regular attractor);

(ii) reduce the lifetime of the transient behavior (forcing the system to reach a regular attractor in the shortest possible time)

To achieve our goals we have to have knowledge of the system dynamics, but not necessarily the equations of motion (in this case the dangerous zone $\mathcal{D}$ and safe set $\mathcal{S}$ can be estimated from the map obtained from experimental time series [8]), which is usually the case in practical systems, and must be able to apply a small temporal change in one of the system parameters.

Additionally, we have introduced practical Lyapunov exponents obtained by averaging over a finite time T (shorter than the transient lifetime τ). Such practical Lyapunov exponents describe the dependence on the initial conditions during the evolution on a strange chaotic repeller.

We hope that the method can be useful in controlling practical dynamical systems which exhibit transient chaos, and that the concept of practical Lyapunov exponents will be useful in the description of transient chaos.

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