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Co-existing attractors of impact oscillator

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Abstract—Co-existing attractors of a mechanical system with impacts and dry friction are discussed in this paper. Our investigations have shown that, in the systems with noise, not all attractors can be reached by system trajectories. © 1998 Elsevier Science Ltd. All rights reserved.

A typical nonlinear system is characterized by the co-existence of different attractors for a fixed set of parameters. This implies that whichever attractor is eventually reached by the system trajectory depends strongly on the initial condition. This phenomenon is called 'multistability' and it is common in various fields of science, such as chemistry [1, 2], optics [3, 4] and neuroscience [5, 6]. Recently, Feudel *et al.* presented the example of the dynamical system with over 1000 attractors [7–9].

The main purpose of this paper is to analyze different types of co-existing attractors in a simple mechanical system. Additionally, we address the problem of the influence of noise on the evolution of different attractors.

As an example, we consider the simple physical system shown in Fig. 1. The mass m_1 is connected to a vibrator giving a sinusoidal force, $F_0 \cos t$ through the spring-damper system with a stiffness coefficient, k_1 and damping coefficient, c_1. The second mass, m_2, is placed on mass m_1 and its movement is limited by two borders, A and B. The motion of mass m_2 on mass m_1 is influenced by friction force, F_t.

In practice, there are many systems whose actions are accompanied by a collision of bodies. Mechanical systems exhibiting impacts, the so-called impact oscillators [10–21], occur widely in technical applications. In many cases, impacts have a positive influence; for instance, in mechanical devices during the thickening and crushing process, working of minerals, impact damping, etc.

The considered model can be described by the following dimensionless equations:

$$\frac{d^2 x_1}{d\tau^2} + b_1 \frac{dx_1}{d\tau} + x_1 + \lambda - b_1 \delta\left(\frac{dx_1}{d\tau} - \frac{dx_2}{d\tau}\right) = (1+\zeta(\tau))\cos \eta\tau$$

$$\frac{d^2 x_2}{d\tau^2} - \frac{\lambda}{\gamma} - \frac{\delta}{\gamma} b_1 \left(\frac{dx_1}{d\tau} - \frac{dx_2}{d\tau}\right) = 0 \tag{1}$$

where: $\Omega_1 = (k_1/m_1)^{1/2}$, $b_1 = c_1/\Omega_1$, $b_2 = c_2/\Omega_1$, $\lambda = F_t/F_0$, $\tau = \Omega_1 t$, $\mu = m_2/m_1$, $\delta = c_2/c_1$ and $f_T = F_t/m_2 g$; g is gravitational acceleration and $\zeta(\tau)$ is a random noise with zero mean and intensity s. In

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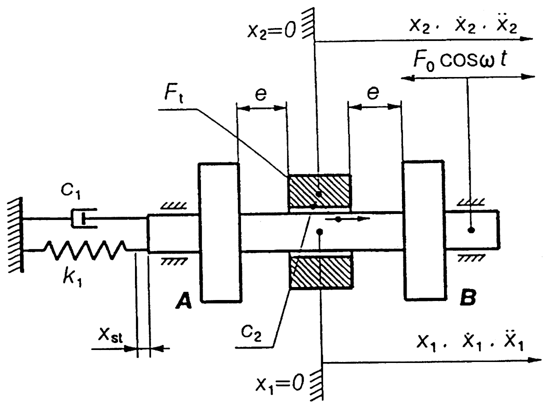

Fig. 1. The model of the system: m_1—primary mass; m_2—additional mass; k_1—spring constant; c_1, c_2—damping constants; x_1, x_2—coordinates of the motion of the masses m_1, m_2; e—static clearance between the masses m_1, m_2; F_t—friction force; F_0—amplitude of the exciting force; ω—angular frequency of the exciting force; Ω_1—natural frequency of the primary system; R—restitution coefficient; $x_{st}=F_0/k_1$ static displacement; $r=e/x_{st}$—relative clearance; $\eta=\omega/\Omega_1$ relative frequency of the exciting force.

order to describe the dry friction force, we considered a linear model [21]. The equation and characteristic of this model are described in [22]. A detailed analysis of a rich bifurcation structure and chaotic behaviour is described in [19, 20].

The bifurcation diagram of the system (1) i.e., the relative displacement, x, versus the control parameter $\eta\in[1.76,2.76]$ plot, for typical system parameters, $\lambda=0.02$, $r=0.8$, $\delta=0.5$, $R=0.6$, $b_1=0.1$ and $\mu=0.693$, is shown in Fig. 2. Narrow intervals of chaotic, periodic and quasi-periodic motion characterize the system's behaviour in each of these regions. The structure of the bifurcation diagrams indicates that the system evolution switches from one attractor to another as the bifurcation parameter is slowly changed in a random, unpredictable fashion.

To understand this phenomenon, the co-existing attractors were identified and their basins of attraction estimated. For most values of $\eta\in[1.96, 2.76]$, four co-existing attractors exist; the periodic one shown in Fig. 3(a), the quasi-periodic one—Fig. 3(b), the different quasi-periodic attractor of Fig. 3(c) and the chaotic one shown in Fig. 3(d).

The basins of attraction of the above-mentioned attractors are shown in Fig. 4(a,b) for $\eta=1.89$ [Fig. 4(a)] and $\eta=2.16$ [Fig. 4(b)]. The cross section of the 5-dimensional phase of the noise-free

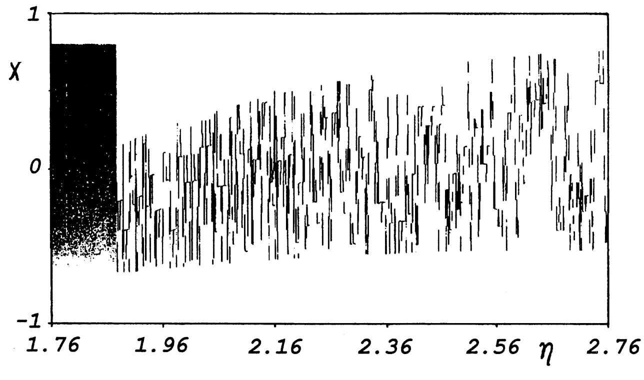

Fig. 2. Bifurcation diagram of the system (1) for typical system parameters: $\lambda=0.02$, $r=0.8$, $\delta=0.5$, $R=0.6$, $b_1=0.1$ and $\mu=0.693$.

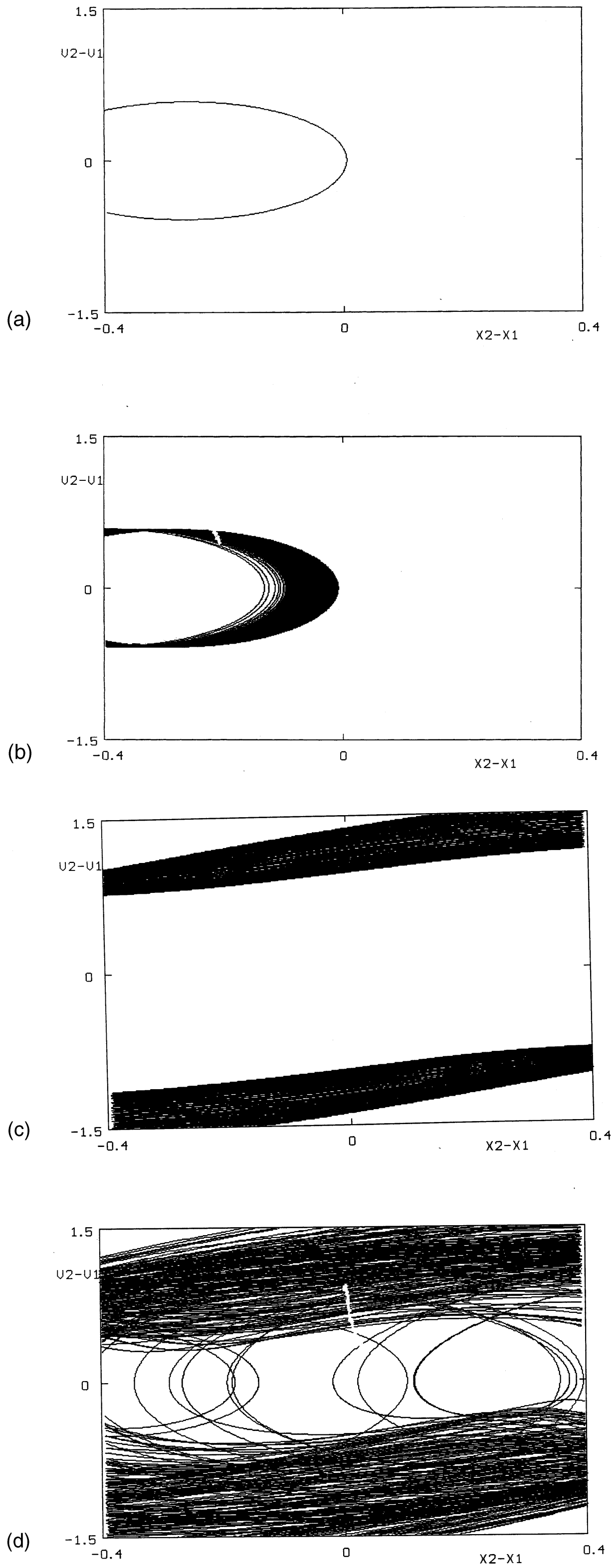

Fig. 3. Different types of co-existing attractors for $\eta = 2.16$; (a) periodic, (b) quasi-periodic, (c) quasi-periodic, (d) chaotic.

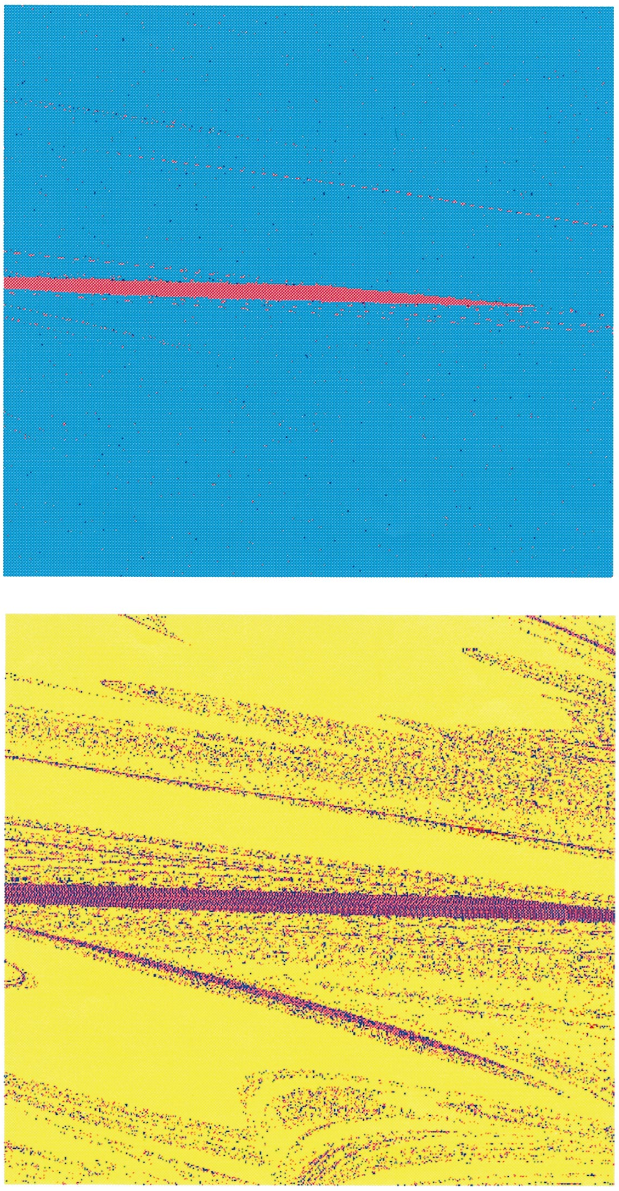

Fig. 4. Basins of attraction of co-existing attractors of Fig. 3; (a) $\eta=1.89$, (b) $\eta=2.16$. The cross section of the 5-dimensional phase of the noise free eqn (1) $(s=0)$, defined as $\Sigma=\{(x_2,\mathrm{d}x_2/\mathrm{d}\tau)|(x_2,\mathrm{d}x_2/\mathrm{d}\tau)\in[-0.4,0.4]\times[-1.5,1.5],$ $x_1=\mathrm{d}x_1/\mathrm{d}\tau=0,\ \tau=2\pi k,\ k=1,2,.\}$ was considered as a set of initial conditions. For the trajectories starting at a point in Σ, the limiting attractor has been determined. The red colour marks the basin of the periodic attractor of Fig. 3(a), the navy-blue marks the basin of the quasi-periodic attractor of Fig. 3(b), the yellow marks the basin of the quasiperiodic attractor of Fig. 3(c) and the blue marks the basin of the chaotic attractor of Fig. 3(d).

eqn (1) $(s=0)$, defined as $\Sigma=\{(x_2,\mathrm{d}x_2/\mathrm{d}\tau)|(x_2,\mathrm{d}x_2/\mathrm{d}\tau)\in[-0.4,0.4]\times[-1.5,1.5],\ x_1=\mathrm{d}x_1/\mathrm{d}\tau=0,$ $\tau=2\pi k,\ k=1,2,.\}$ was considered as a set of initial conditions. For the trajectories starting at a point in Σ, the limiting attractor has been determined. The red colour marks the basin of the periodic attractor of Fig. 3(a), the navy blue marks the basin of the quasi-periodic attractor of Fig. 3(b), the yellow marks the basin of the quasiperiodic attractor of Fig. 3(c) and the blue marks the basin of the chaotic attractor of Fig. 3(d). Figure 4 shows that the system's behaviour is strongly dependent on small changes of the initial conditions. We can observe mostly chaotic motion and three attractors co-exist in phase space.

The analysis of Fig. 4(a,b) shows that the basins of some attractors are so small that random noise prevents trajectories from reaching them. For example, for $s=0.01$ and $\eta=1.89$, only periodic and chaotic attractors of, respectively, Fig. 3(a) and Fig. 3(b) can be reached by the system and, for larger noise intensity, $s=0.1$, only the chaotic attractor is possible. Global bifurcations of basin boundaries, resulting in the size changes of particular basins, result in the trajectories jumping from one attractor to another. These jumps explain the structure of the bifurcation diagram of Fig. 2.

Co-existing attractors were found to be common in the mechanical system with impacts and dry friction. Our investigations have shown that when the noise exists in the system, as it does in the experiments, some attractors cannot be reached as their basin of attraction is too small and the noise forces the trajectories out of them towards other attractors with larger basins of attraction. Similar results have been obtained by Kraut *et al.* [23]. Additionally, it was shown that, even with very small noise (equal to that of numerical errors), the position of the trajectory on the attractor at the moment when the bifurcation parameter is being changed can affect the result of the bifurcation.

REFERENCES

1. Marmillot, P., Kaufmann, M., Hervagault, J. F. *J. Chem. Phys.*, 1991, **95,** 1206.
2. Laplante, J. P., Erneux, T. *Physica*, 1992, **A188,** 89.
3. Hammel, M., Jones, C. K. R. T., Moloney, J. V. *J. Opt. Soc. Am. B*, 1985, **2,** 552.
4. Brambilla, M., Lugiato, L. A., Penna, V. *Phys. Rev.*, 1991, **A43,** 5114.
5. Schiff, S., Jerger, K., Duong, D. H., Chang, T., Spano, M. L., Ditto, W. L. *Nature*, 1994, **370,** 615.
6. Foss, J., Longin, A., Mensour, B., Milton, J. *Phys. Rev. Lett.*, 1996, **76,** 708.
7. Feudel, U., Grebogi, C., Hunt, B., Yorke, J. A. *Phys. Rev.*, 1996, **E54,** 71.
8. Feudel, U., Grebogi, C. *Chaos*, 1997, **7,** 597.
9. Feudel, U., Poon, L., Grebogi, C., Yorke, J. A. *Chaos, Solitons and Fractals*, 1998, **9,** 171.
10. Bishop, R. *Phil. Trans. R. Soc. London*, 1994, **A347,** 341–351.
11. Thompson, J. M. T., Ghaffari, R. *Phys. Lett.*, 1982, **A91,** 5–8.
12. Shaw, S., Holmes, P.J. *J. Sound Vib.*, 1983, **90,** 129–155.
13. Isomaki, H., Boehm, J., Raty, R. *Phys. Lett.*, 1985, **A107,** 343–346.
14. Peterka, F., Vacik, J. *J. Sound Vib.*, 1992, **154,** 95–115.
15. Nordmark, A. B. *J. Sound Vib.*, 1991, **145,** 279–297.
16. Chin, W., Ott, E., Nusse, H., Grebogi, C. *Phys. Lett.*, 1995, **A201,** 279–297.
17. Foale, S., Bishop, S. R. *Trans. R. Soc. London*, 1992, **A338,** 547–556.
18. Blazejczyk, B., Kapitaniak, T., Wojewoda, J., Barron, R. *J. Sound Vib.*, 1994, **178,** 272–275.
19. Blazejczyk, B., Analysis of nonregular motion in the system with impacts. Ph.D. thesis, Technical University of Lodz, 1995.
20. Blazejczyk-Okolewska, B., Kapitaniak, T. *Chaos, Solitons and Fractals*, 1996, **7(9),** 1455–1459.
21. Den Hartog, J. P. *Trans. ASME*, 1931, **53,** 107–115.
22. Poop, K. and Stelter, P., Nonlinear oscillations of structures induced by dry friction. In *Non-linear Dynamics in Engineering Systems*, ed. Schiehlen, W. Springer, New York, 1990.
23. Kraut, S., Feudel, U., Grebogi, C. (in preparation).