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Practical riddling in mechanical systems

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Abstract

We have shown that in practical systems the existence of co-existing attractors with unequal basins with fractal boundaries can lead to the uncertainties similar to the uncertainty introduced by riddled basins in coupled systems. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

Recently riddled basins have been found to be characteristic for higher-dimensional coupled systems [1–4] as they have been observed both numerically and experimentally. The existence of riddled basins has introduced a new type of uncertainty in dynamical systems, i.e. having the system operating on the desired attractor we cannot be sure if this system will operate on the same attractor after small perturbation to its trajectory.

In this paper we give evidence that in practical systems the existence of co-existing attractors unequal basins with fractal boundaries can lead to the uncertainties similar to the uncertainty introduced by riddled basins in coupled systems.

The outline of this paper is as follows. In Section 2 we recall some fundamental definitions of stability of chaotic attractors and describe attractors with riddled basins. The simple mechanical system with impacts and dry friction is investigated in Section 3. We identify co-existing attractors and their basins and show that when the small noise is added into the systems some of these attractors cannot be reached. Finally we summarize our results in Section 4.

2. Theoretical riddling

Let us recall the fundamental properties of chaotic attractors with riddled basins. In our description we use the system of two coupled chaotic maps which can be considered as a prototype of higher-dimensional dynamical systems.

Two identical chaotic systems $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n), x, y \in \mathbb{R}$, evolving on asymptotically stable attractor A, when one-to-one coupling

$$\begin{aligned} x_{n+1} &= f(x_n) + d_1(y_n - x_n) \\ y_{n+1} &= f(y_n) + d_1(x_n - y_n) \end{aligned} \tag{1}$$

is introduced can be synchronized for some values of $d_{1,2} \in \mathbb{R}$, i.e., $|x_n - y_n| \to 0$ as $n \to \infty$ [1–10].

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In the synchronized regime the dynamics of the coupled system (1) is restricted to one-dimensional invariant subspace $x_n = y_n$, so the problem of synchronization of chaotic systems can be understood as a problem of stability of one-dimensional chaotic attractor A in two-dimensional phase space [12,13]. The basin of attraction $\beta(A)$ is the set of points whose ω-limit set contained in A. In Milnor's definition [11] of an attractor the basin of attraction does not need to include the whole nieghbourhood of the attractor, i.e., we say that A is a weak Milnor attractor if $\beta(A)$ has positive Lebesgue measure. For example, a riddled basin [12–15] which has recently been found to be typical for a certain class of dynamical systems with an invariant subspace $x = y$ like for example Eq. (1), has positive Lebesgue measure but does not contain any neighbourhood of the attractor. In this case an attractor A is transversely stable in the invariant subspace $x_n - y_n$, but its basin of attraction $\beta(A)$ may be a fat fractal so that any neighbourhood of the attractor intersects the basin with positive measure, but may also interest the basin of another attractor with positive measure.

The dynamics of the systems (1) is described by two Lyapunov exponents. One of them describes the evolution on the invariant manifold $x = y$ and is always positive. The second exponent characterizes the evolution transverse to this manifold and it is called transversal. If the transversal Lyapunov exponent is negative, A is an attractor in the weak Milnor sense.

When the transversal Lyapunov exponent is negative, A is a locally riddled attractor, i.e., there is a neighbourhood U of A such that in any neighbourhood V of any point in A, there is a set of points in $V \cap U$ of positive measure which leave U in a finite time. The trajectories which leave neighbourhood U can either go the other attractor (attractors) or after a finite number of iterations be diverted back to A. If there is a neighbourhood U of A such that in any neighbourhood V of any point in A, there is a set of points in V \cap U of positive measure which leaves U in finite time and goes to another attractor, then the basins of A are globally riddled.

3. Practical riddling

The phenomenon of riddled basins described in the previous section seems to be very difficult to be observed in mechanical engineering systems as for example the condition of coupled systems to be identical (necessary for coupled system to have invariant manifold) is nearly impossible to be fulfilled. However, we will shortly argue that the uncertainty similar to the one caused by riddled basins can occur in high-dimensional systems.

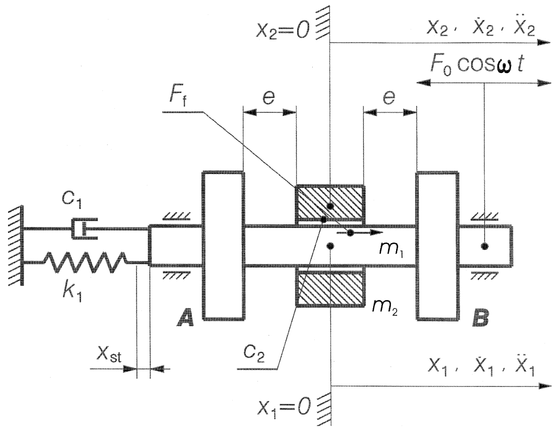
Let us consider the simple physical system shown in Fig. 1. The mass m_1 is connected to a vibrator giving sinusoidal force $F_0 \cos \omega t$ through the spring-damper system with stiffness coefficient k_1 and damping coefficient c_1. The second mass m_2 is placed on mass m_1 and its movement is limited by two borders A and B. The motion of mass m_2 on mass m_1 is influenced by friction force F_t.

The considered model can be described by the following dimensionless equations:

$$\begin{aligned} X_1^{||} + b_1 X_1^{|} + X_1 + \lambda - b_1 \delta (X_1^{|} - X_2^{|}) &= \cos \eta\tau \\ x_2^{||} - (\lambda/\mu) - (\delta/\mu) b_1 (X_1^{|} - X_2^{|}) &= 0, \end{aligned} \tag{2}$$

where: $\Omega_1 = (k_1/m_1)^{1/2}$, $b_1 = c_1/\Omega_1$, $b_2 = c_2/\Omega_1$, $X^{|} = \mathrm{d}X/\mathrm{d}\tau$, $X^{||} \mathrm{d}^2 X/\mathrm{d}\tau^2$, $\lambda = F_t/F_0$, $\tau = \Omega_1 t$-time transformation, $\mu = m_2/m_1$, $\delta = c_2/c_1$, $f_T = F_t/m_2 g$ and g is gravitational acceleration. In order to describe the dry friction force we have considered a linear model [19]. The equation and characteristic behaviour of this model have been investigated in e.g. [16,17].

The bifurcation diagram of the system (2) the relative displacement x vs. control parameter $\eta \in [1.76, 2.76]$ for typical system parameters $\lambda = 0.02, r = 0.8, \delta = 0.5, R = 0.6, b_1 = 0.1$ and $\mu = 0.693$ is presented in Fig. 2. This figure shows the complicated structure which is caused by the jumps of system trajectory from one of three or four co-existing attractors of different type (periodic, chaotic and two different quasi-periodic attractors can be observed [18]) to another as the bifurcation parameter is slowly changed. This jumps occur in random unpredictable fashion and as a result of them one cannot predict on which attractor the system will evolve after small perturbation of it.



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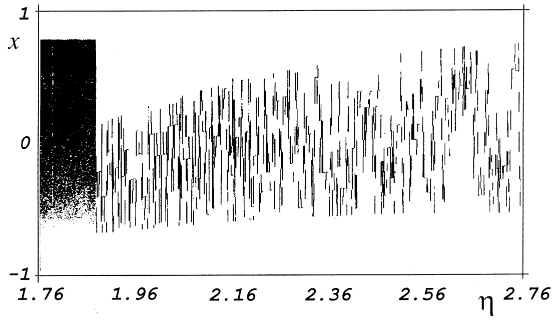

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Fig. 1. The model of the system: m_1: primary mass, m_2: additional mass, k_1: spring constant, c_1, c_2: damping constants, x_1, x_2: coordinates of the motion of the masses m_1, m_2, e: static clearance between the masses m_1, m_2, F_t: friction force, F_0: amplitude of the exciting force, ω: angular frequency of the exciting force, Ω_1: natural frequency of the primary system, R: restitution coefficient, $x_{st} = F_0/k_1$: static displacement, $r = e/x_{st}$: relative clearance, $\eta = \omega/\Omega_1$: relative frequency of the exciting force.

Fig. 2. Bifurcation diagram of the system (2) for typical system parameters: $\lambda = 0.02, r = 0.8, \delta = 0.5, R = 0.6, b_1 = 0.1$ and $\mu = 0.693$.

The basins of attraction of the above-mentioned attractors are shown in Fig. 3(a) and (b) for $\eta = 1.89$ (Fig. 3(a)) and $\eta = 2.16$ (Fig. 3(b)). The cross-section of the 5-dimensional phase Eq. (2), defined as $\Sigma = \{(X_2, X_2^| = \mathrm{d}X_2/\mathrm{d}\tau) | (X_2, X_2^| = \mathrm{d}X_2/\mathrm{d}\tau) \in [-0.4, 0.4] \times [-1.5, 1.5],\ X_1 = X_1^| = \mathrm{d}X/\mathrm{d}\tau = 0, \tau = 2\pi k,\ k = 1, 2, \ldots\}$ was considered as a set of initial conditions. For the trajectory starting at a point in Σ, the limiting attractor has been determined. The red colour marks the basin of the periodic attractor, the navy-blue marks the basin of the quasi-periodic attractor one, the yellow marks the basin of different quasi-periodic attractor and the blue marks the basin of the chaotic attractor. This figure shows that the system's behaviour is a strongly dependent on small changes of the initial conditions. We can observe mostly chaotic motion and three attractors co-exist in phase space.

The analysis of Fig. 3 shows that the basins of some attractors are so small that random noise prevents trajectories from reaching them. For example, for small noise intensity and $\eta = 1.89$ only periodic and chaotic attractors can be reached by the system and for larger noise intensity only the chaotic attractor is possible. Global bifurcations of basin boundaries, resulting in the size changes of particular basins, result in

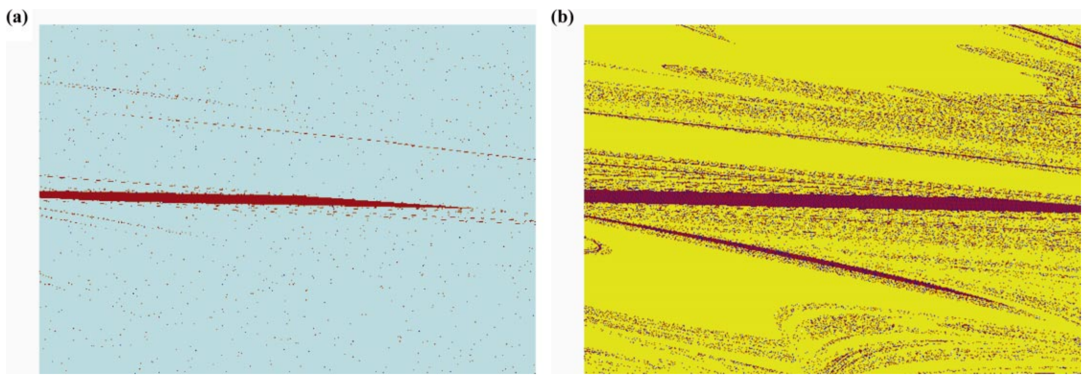

Fig. 3. Basins of attraction of co-existing attractors; (a) $\eta = 1.89$, (b) $\eta = 2.16$. The cross section of the 5-dimensional phase Eq. (2), defined as $\Sigma = \{(X_2, X_2^| = \mathrm{d}X_2/\mathrm{d}\tau) | (X_2, X_2^| = \mathrm{d}X_2/\mathrm{d}\tau) \in [-0.4, 0.4] \times [-1.5, 1.5],\ X_1 = X_1^| = \mathrm{d}X/\mathrm{d}\tau = 0, \tau = 2\pi k,\ k = 1, 2, \ldots\}$ was considered as a set of initial conditions. For the trajectory starting at a point in Σ, the limiting attractor has been determined. The red colour marks the basin of the periodic attractor, the navy blue marks the basin of the quasi-periodic attractor one, the yellow marks the basin of different quasi-periodic attractor and the blue marks the basin of the chaotic attractor.

the trajectories jumping from one attractor to another. These jumps explain the structure of the bifurcation diagram of Fig. 2. A detailed analysis of Fig. 3 is described in [18].

4. Conclusions

We have shown that in practical systems the existence of co-existing attractors with unequal basins with fractal boundaries can lead to the sudden unexpected jumps of the system trajectory from one attractor to another. These jumps cannot be predicted and are present no matter how small are noise level in our systems. Dynamical uncertainty introduced by these jumps is similar to the uncertainty introduced by riddled basins in coupled systems.

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