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Partially nearly riddled basins in systems with chaotic saddle

Tomasz Kapitaniak

Division of Dynamics, Technical University of Lodz, Stefanowskiego 1/15, 90-924 Lodz, Poland

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Abstract

We show that chaotic attractors can have partially nearly riddled basins of attraction, i.e., basins which consist both of large open sets and a set in which small open sets which belong to the basins of different attractors are intermingled. We argue that such basins are robust for systems with the chaotic saddle located between at least two attractors and in the presence of noise cause the uncertainties similar to those implied by riddled basins.

Over the last decade the field of coupled chaotic systems has advanced considerably due to the extensive studies of chaos synchronization phenomenon [1–12]. Consider two identical chaotic systems $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n),\ x, y \in \mathbb{R}$, evolving on asymptotically stable chaotic attractor A. It has been shown that when some kind of coupling

$$\begin{aligned} x_{n+1} &= f(x_n) + \epsilon_1 g_1(x_n, y_n), \\ y_{n+1} &= f(y_n) + \epsilon_2 g_2(x_n, y_n), \end{aligned} \tag{1}$$

where $\epsilon_{1,2} \in \mathbb{R}$ and $g_{1,2}(x_n, y_n)$ is a real function describing coupling with a property that $g_1 = g_2 = 0$ for $x_n = y_n$, is introduced, x- and y-systems can be synchronized for some ranges of $\epsilon_{1,2}$, i.e., $|x_n - y_n| \to 0$ as $n \to \infty$.

In the synchronized regime the dynamics of the coupled system (1) is restricted to one-dimensional invariant subspace $x_n = y_n$, so the problem of synchronization of chaotic systems can be understood as a problem of stability of one-dimensional chaotic attractor A in two-dimensional phase space [13–22].

The basin of attraction $\beta(A)$ is the set of points whose ω-limit set is contained in A. In Milnor's definition [23] of an attractor, the basin of attraction need not include the whole neighborhood of the attractor, i.e., we say that A is a weak Milnor attractor if $\beta(A)$ has a positive Lebesgue measure. For example, a riddled basin [13–22] which has recently been found to be typical for a certain class of dynamical systems with one-dimensional invariant subspace (like $x_n = y_n$ in the example (1)) has positive Lebesgue measure but does not contain any neighborhood of the attractor. In this case the basin of attraction $\beta(A)$ may be a fat fractal so that any neighborhood of the attractor intersects the basin with positive measure, but may also intersect the basin of another attractor with positive measure.

The dynamics of system (1) is described by two Lyapunov exponents. One of them describes the evolution on the invariant manifold $x = y$ and is always positive. The second exponent characterizes evolution transverse to this manifold and it is called transversal. If the transversal Lyapunov exponent is negative, the set A is an attractor, at least in the weak Milnor sense.

E-mail address: tomaszka@ck-sg.p.lodz.pl (T. Kapitaniak).

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When the transversal Lyapunov exponent is negative, and there exist trajectories in the attractor A, which are transversally repelling, A is a weak Milnor attractor with *locally riddled* basin, i.e., there is a neighborhood U of A such that in any neighborhood V of any point in A, there is a set of points in $V \cap U$ of positive measure which leaves U in a finite time. The trajectories which leave neighborhood U can either go to the other attractor (attractors) or after a finite number of iterations be diverted back to A. If there is a neighborhood U of A such that in any neighborhood V of any point in U, there is a set of points of positive measure which leave U and go to the other attractor (attractors), then the basin of A is *globally riddled*.

As the conditions of the definition of riddled basins are very difficult to be fulfilled in practical engineering systems, a definition of practical riddling has been introduced [24]. In practical riddling the neighborhood U of the attractor A belongs to its basin $\beta(A)$. $\beta(A)$ is so small that any real perturbation forces the trajectory out of U to the region of the phase space where the basin's structure is so complex that the fate of the trajectory, i.e., attractor A or co-existing attractor B (attractors $B, C, \ldots$) cannot be predicted. Systems with practically riddled basins in the presence of noise with given finite level have the same property as sytems with riddled basins.

The problem of chaos synchronization is not the only theoretical problem studied in coupled chaotic systems. The other interesting problem is chaos–hyperchaos transition [24–29], i.e., transition from the attractor with one positive Lyapunov exponent to the attractor with at least two positive exponents. In [29] some similarities between this transition and the phenomena associated with the loss of chaos synchronization have been shown. One of these similarities is the occurrence of a locally riddled basin of the chaotic attractor close to the transition to hyperchaos.

In this paper we show that the other type of basin of attraction, the basin which contains open sets but with a peculiar property, can occur in coupled systems with at least two co-existing attractors. This peculiarity is caused by the existence of a region in the phase space in which small open sets which belong to the basins of different co-existing attractors are intermingled. These sets are open as preimages of the large open sets located in the neighborhoods of the attractors (for example its immediate basins), but on the other hand so small that in the relatively small (in comparison with the noise level) neighborhood of any point in them there are points which belong to the basin of different attractor. We give evidence that such basins, which we called *partially nearly riddled*, are robust for systems with the chaotic saddle located between at least two attractors.

As an example we consider the dynamics of two coupled logistic maps:

$$\begin{aligned} x_{n+1} &= \lambda - x_n^2 + \epsilon(x_n^2 - y_n^2), \\ y_{n+1} &= \lambda - y_n^2 + \epsilon(y_n^2 - x_n^2), \end{aligned} \tag{2}$$

where x_n, y_n are the dynamical variables, and λ and the coupling coefficient ϵ are the controlling parameters of the system.

The primary reason that we choose to illustrate our results using two-dimensional maps is that for such systems, there exists a procedure, the proper-interior-maximum triple (PIM-triple) procedure [30], for computing an arbitrarily long trajectory on a chaotic saddle with high precision. We are not aware of any procedure that can be utilized to compute trajectories on chaotic saddles in higher dimensions.

Our map (1) is noninvertable and its attractor presents an invariant chaotic set which is enclosed in the so-called *chaotic area* (according to the terminology of Mira [31]). This chaotic area is an invariant region of the two-dimensional phase space bounded typically by a finite number of segments of the so-called *critical lines*, which are obtained from the iterations of the set $L_0 = \{(x,y) : DF(x,y) = 0\}$ i.e., curves where Jacobian DF vanishes, where F is the map given by (1).

In our numerical study we take $\lambda = 1.56$ and consider $\epsilon \in [0.036, 0.042]$. At these parameter values, system (2) has either two co-existing chaotic attractors or one hyperchaotic attractors as shown in Figs. 1(a) and (b) inside the chaotic area. Fig. 1(a) shows chaotic attractors A and B for $\epsilon = 0.0390$ and Fig. 1(b) presents hyperchaotic attractor H for $\epsilon = 0.0385$. Chaotic attractors are characterized by one positive Lyapunov exponent while hyperchaotic ones by two positive Lyapunov exponents for typical trajectories in the phase sapce. At $\epsilon = 0.03854$, one observes the transition from chaos to hyperchaos.

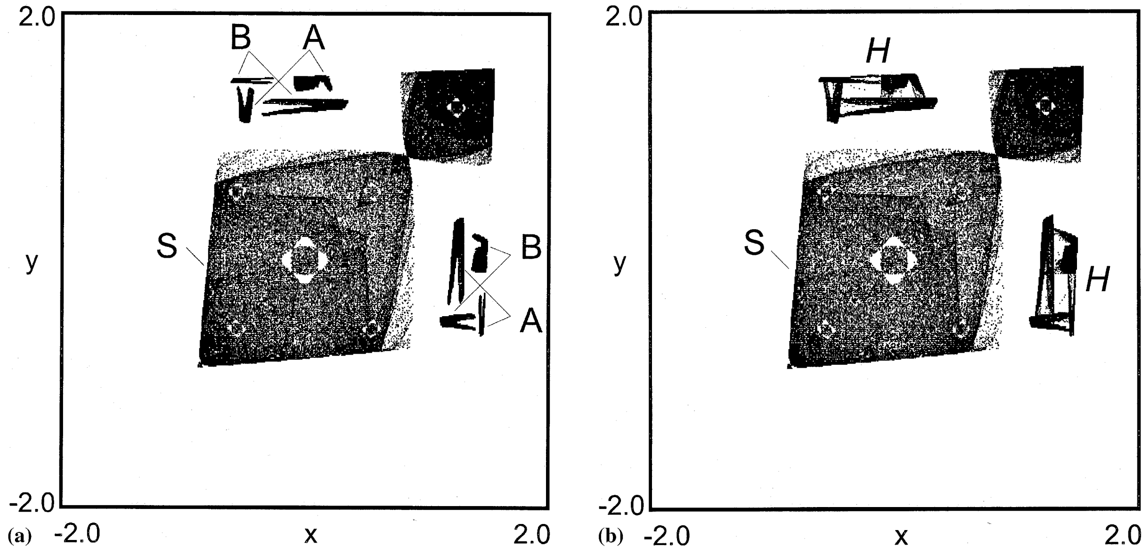

Fig. 1. Chaotic (A and B), hyperchaotic (H) attractors and chaotic saddle S of the map (2) for $\lambda = 3.8$; (a) $\epsilon = 0.0390$, (b) $\epsilon = 0.0385$.

Additionally to attracting sets A and B or H for $\epsilon \in [0.0383, 0.0420]$ there exists a chaotic nonattracting set – chaotic saddle S. Typical trajectory on this chaotic saddle calculated by PIM-tripe algorithm is characterized by two positive Lyapunov exponents. Although system (2) has symmetry, its attractors shown in Figs. 1(a) and (b) are not located at the invariant manifold, so this system can be used for the description of the general properties of basins of attraction in the neighborhood of chaos–hyperchaos transition.

Consider basins of attraction characteristic for the chaotic attractors A and B. An example of such basins for $\epsilon = 0.04$ is shown in Fig. 2(a) and at the enlargement in Fig. 2(b). Basins of attractors A (green) and B (red) are indicated, respectively, in blue and pink while navy blue indicates the basin of the attractor at infinity. These figures clearly show that basins of both chaotic attractors are characterized both by open sets (blue or pink) and a nearly riddled set (where blue is intermingled with pink). In this set small open sets which belong to the basins of attractors A and B are intermingled. Nearly riddled set is bounded by critical curves. This new type of basins of attraction we called as *partially nearly riddled.* Partially riddled basins are robust for $\epsilon > 0.0854$, i.e. before chaos–hyperchaos transition but they disappear at this transition as can be seen in Fig. 2(c) (blue indicates the basin of hyperchaotic attractor H and navy blue indicates the basin of the atractor at infinity).

It should be mentioned here that inside the considered ϵ-interval, various periodic attractors with very small basins can exist in different small ϵ-subintervals. As they are not important in the creation of partially nearly riddled basins they are not discussed here.

The creation of partially nearly riddled basins is connected with the existence of the chaotic saddle and at least two attractors in the phase space. As it can be seen in Figs. 1(a) and 2(a) and (b), the chaotic saddle S is located between open sets which create basins of two co-existing attractors A and B. Trajectories evolving on the saddle after some time have to go to one of the attractors A or B. Due to the symmetry in the system, attractors A and B are equally probable as a final goal of any trajectory from the saddle S. Relatively small (but not infinitely small) noise can change the fate of such a trajectory. This uncertainty creates the partially nearly riddled set in the phase space, where the chaotic saddle or its preimages exist. In the case of Figs. 1(b) and 2(c), the chaotic saddle S still exists but there is only one attractor H at the chaotic area of system (2), so the trajectories originally evolving on S have to approach the attractor H and the nearly riddled set is not created in the phase space.

Partially nearly riddled basin can also be created in the case when only one attractor exists inside the chaotic area, after the destruction of this area in the boundary crisis of the chaotic saddle with the boundary of the attractor in infinity. In this case the basin of the attractor in infinity invades the original attractor initially located inside the chaotic area.

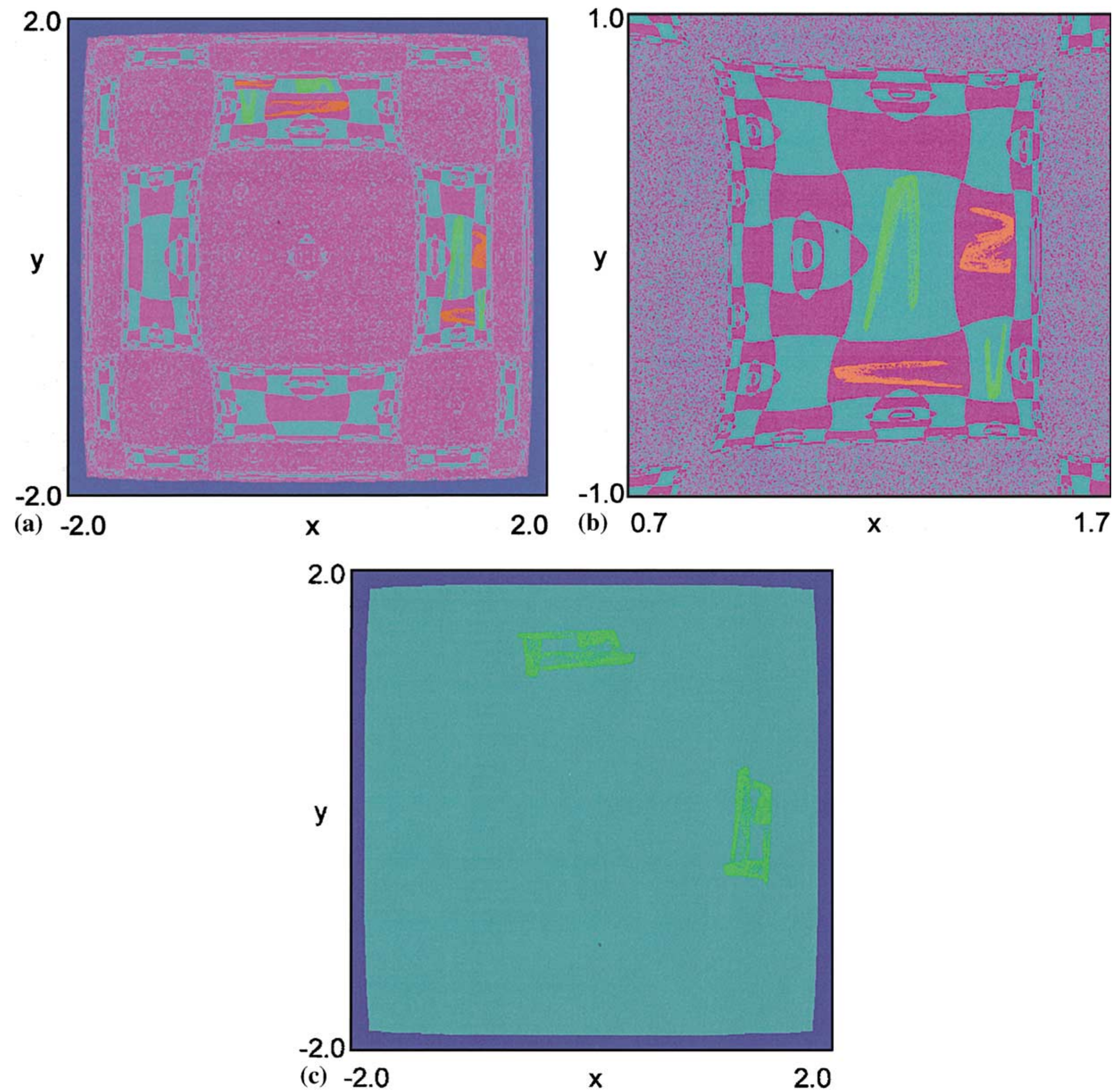

Fig. 2. Basin of attraction of chaotic (A and B), hyperchaotic (H) attractors and chaotic saddle S of the map (2) for $\lambda = 3.8$; (a,b) $\epsilon = 0.0390$, basins of attractors A (green) and B (red) are indicated, respectively, in blue and pink while navy blue indicates the basin of the attractor at infinity, (c) $\epsilon = 0.0385$, blue indicates the basin of hyperchaotic attractor H (green) and navy blue indicates the basin of the attractor at infinity.

In summary, we describe a new class of basins of attraction, namely the partially nearly riddled basin which consists of both open and nearly riddled sets. We outline a mechanism for the occurrence of such basins in systems with co-existing attractors and a chaotic saddle. The existence of partially nearly riddled basins analyzed in this paper is a high-dimensional phenomenon that can be expected in systems such as coupled map lattices or coupled ordinary differential equations, which arise naturally when one discretizes a nonlinear partial differential equation.

Acknowledgements

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