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Bubbling and riddling of higher-dimensional attractors

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Abstract

We analyze the bifurcation in which one of the unstable periodic orbits embedded in a higher-dimensional chaotic attractor becomes unstable transversely to the attractor. The existence of such local transversal instability may cause the bubbling of the attractor in the invariant manifold or it may cause the riddling of the basin of attraction.

1. Introduction

Unstable periodic orbits (UPOs) constitute one of the most basic invariants of a dynamical system [1]. The infinite number of UPOs embedded in a chaotic set provides the skeleton of the attractor and it allows for the characterization and the estimation in a fundamental way of many dynamical invariants such as the natural measure, the spectra of Lyapunov exponents and the fractal dimension [2]. UPOs play a fundamental role in the mechanism of destabilization of the chaotic attrator localized in some symmetric invariant manifold and it is responsible for the dynamics of phenomena such as riddling of the basin of attraction and bubbling of the chaotic attractor [3]. Recently, UPOs have also been used in the description of higher-dimensional dynamical phenomena of chaos–hyperchaos transition (i.e., transition from the attractor characterized by one positive Lyapunov exponent to the attractor characterized by at least two positive exponents) [4–6]. It has been shown that the chaos–hyperchaos transition, as well as the blowout bifurcation, is mediated by an infinite number of UPOs which become unstable in at least two directions in the neighborhood of the transition point. The simultaneous existence of UPOs with different number of unstable direction gives rise to a new kind of non-hyperbolicity known as unstable dimension variability [7] and it may give a possible dynamic mechanism for the smooth transition through zero of the second Lyapunov exponent.

In this paper, we argue that the phenomena characteristic of one-dimensional attractors located on the invariant manifold, like riddling and bubbling, can be generalized to higher-dimensional attractors. We show that the riddling, which occurs after the appearance of the first UPO with more than one unstable direction in the chaotic attractor on the invariant manifold allows for the growth of the attractor by the bursting along the new unstable direction. We point out that this is a typical way by which higher-dimensional attractors grow.

The paper is organized as follows. In Section 2, we recall some fundamental properties of the stability of the one-dimensional attractors located on the invariant manifold. Section 3 describes the mechanism of bubbling of higher-dimensional attractors. An example illustrating bubbling mechanism is shown in Section 4. Finally, we summarize our results in Section 5.

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2. Riddling of one-dimensional attractors

Two identical chaotic systems $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$, $x, y \in \mathscr{R}$ evolving on an asymptotically stable chaotic attractor A, when coupled as

$$\begin{aligned} x_{n+1} &= f(x_n) + d_1(y_n - x_n), \\ y_{n+1} &= f(y_n) + d_2(x_n - y_n), \end{aligned} \tag{1}$$

can be synchronized for some ranges of $d_{1,2} \in \mathscr{R}$, i.e., $|x_n - y_n| \to 0$ as $n \to \infty$ [8]. In the complete synchronized regime, the dynamics of the coupled system (1) is restricted to one-dimensional invariant subspace $x_n = y_n$, so the problem of synchronization of chaotic systems can be understood as a problem of stability of one-dimensional chaotic attractor A in two-dimensional phase space.

The basin of attraction $\beta(A)$ is the set of points whose ω-limit set is contained in A. In Milnor's definition [9] of an attractor, the basin of attraction needs not to include the whole neighborhood of the attractor, i.e., we say that A is a weak Milnor attractor if $\beta(A)$ has positive Lebesgue measure. For example, a riddled basin [10], which has recently been found to be typical for a certain class of dynamical systems with one-dimensional invariant subspace (like $x_n = y_n$ in the example (1)), has positive Lebesgue measure but does not contain any open neighborhood. In this case, the basin of attraction $\beta(A)$ may be a fat fractal so that any neighborhood of the attractor intersects its own basin with positive measure, but it intersects the basin of another attractor also with positive measure.

Dynamics of the system (1) is described by two Lyapunov exponents. One of them describes the evolution on the invariant manifold $x = y$ and is always positive. The second exponent characterizes evolution transverse to this manifold and it is called the transversal exponent. If the transversal Lyapunov exponent is negative, the set A is an attractor, at least in the weak Milnor sense.

It may happen that, though the transversal Lyapunov exponent is negative, there exist trajectories in the attractor A which are transversely repelling. In this case A is a Milnor attractor with *locally riddled* basin, i.e., there is an open neighborhood U of A such that in any neighborhood V of any point in A, there is a set of points in $V \cap U$ of positive measure which leaves U in a finite time. The trajectories which leave the neighborhood U can either go to the other attractor (attractors) or after a finite number of iterations be diverted back to A. The latter case is also known as *bubbling* of attractor A.

Transition from asymptotically stable attractor to the Milnor attractor with riddled basin occurs via riddling bifurcation [11] in which one of the UPO's (say O_1) embedded in the attractor A becomes transversely unstable. (It becomes an unstable repelling node.) This transverse instability allows the trajectories near the attractor A to escape. In the neighborhood of O_1, two tongues of points that do not belong to the basin of attraction of the attractor in the invariant manifold are developed. Moreover, each preimage of O_1 also develops such tongues. Since preimages of O_1 are dense in the invariant manifold, an infinite number of tongues is created simultaneously.

3. Riddling bifurcation of higher-dimensional attractors

In the previous paper [6], we studied the dynamical system given by a dissipative map $u_{n+1} = f(a, u_n)$, where $u \in \mathbb{R}^2$ and $a \in \mathbb{R}$. In such system, due to the stretching and folding mechanism, one can observe attractors with one or two positive Lyapunov exponents. Generally, if such a map is N-dimensional ($u \in \mathbb{R}^N$) one can observe attractors with N positive Lyapunov exponents. We assumed that the system evolved on the chaotic attractor A (i.e., with one positive Lyapunov exponent) and allowed the control parameter to vary slowly in such a way that the second Lyapunov exponent became positive and thus the attractor A became hyperchaotic. We gave evidence [6] that the bifurcations of UPO which are characteristic for the chaos–hyperchaos transition are typically stretching (spreading) in a given control parameter interval, and that the transition mechanism has the same characteristic features as the blowout bifurcation of the attractors located in an invariant manifold in systems with symmetry [3]. It was shown that the transition to hyperchaos is initiated when a repeller arises in the attractor through one of the following situations: (i) some usually low-period periodic orbit O_1, embedded in the chaotic attractor A undergoes a saddle-repeller bifurcation, (ii) a repelling node in the attractor appears in a saddle-node bifurcation, (iii) the repeller (unstable node or focus) originally located off the attractor is absorbed by the expanding attractor.

In the present work, we consider the chaotic attractor A located in a three-dimensional phase space, as shown in Fig. 1, and we denote one of the UPOs embedded in it by O_1. Before the bifurcation, O_1 has stable S_1 and unstable U_1 manifolds located on the attractor A and stable manifold S_2 transverse to A (Fig. 1(a)) [12]. After the bifurcation the manifold S_2 transverse to A becomes unstable. In Fig. 1(b) it is denoted by U_2.

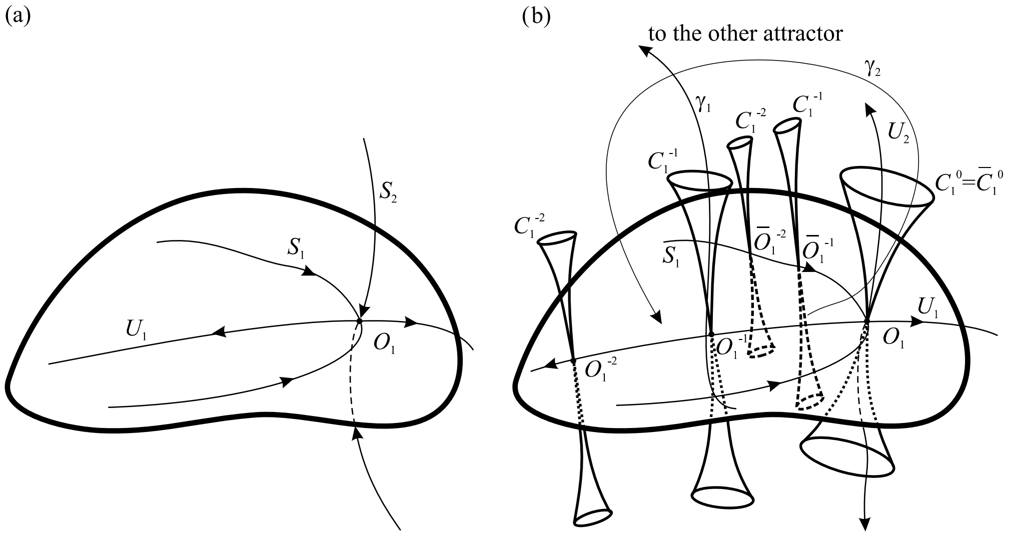

Fig. 1. The mechanism which allows bubbling of higher-dimensional attractors: (a) before the riddling bifurcation,(b) after the riddling bifurcation.

If the considered map is non-invertible appearance of the first UPO with more than one unstable direction on the attractor A creates the tongues $C_1, C_1^{-1}, C_1^{-2}, \ldots$ anchored respectively at O_1 and at all preimages of O_1 (denoted by $O_1^{-1}, O_1^{-2}, \ldots$) on the attractor A with such a property that all points in these sets leave the neighborhood of A. (In any open neighborhood U of A there is a positive measure set of points which leave this neighborhood.) The system trajectories entering the neighborhood of O_1 or neighborhoods of all preimages of O_1 located on the closure of the unstable manifold U_1 on the attractor A leave the attractor along the unstable manifold (U_2) which is transverse to A. The trajectory which leaves the attractor A, could be asymptotic to the other attractor B (trajectory γ_1 in Fig. 1(b)) and in this case the basin of attractor A is riddled.

If the considered system is invertible there exists orbit $\{\bar{O}_1^n\}_{n=-\inf}^{n=\inf} \in S_1 \in A$ approaching O_1 as $t \to \inf$. At the points $\overline{O}_1^n$ the second sequence of tongues $\overline{C}_1, \overline{C}_1^{-1}, \overline{C}_1^{-2}, \ldots$ is anchored as can be seen in Fig. 1(b). The fate of any trajectory entering these tongues is the same as described for tongues $C_1, C_1^{-1}, C_1^{-2}, \ldots$ in the case of non-invertible system.

Riddling of m-dimensional attractor A ($m > 1$) can be defined as follows; *basin of attraction $\beta(A)$ of attractor A is called riddled if there exists an infinite set $R \subset A$ with such a property that in any open neighborhood of R there exist points which do not belong to $\beta(A)$*. In the example shown in Fig. 1, the set R is given by $O_1 \cup O_1^{-1} \cup O_1^{-2} \cup \cdots$ or $\overline{O}_1 \cup \overline{O}_1^{-1} \cup \overline{O}_1^{-2} \cup \cdots$ As the basin of attraction $\beta(A)$ becomes riddled, after the appearance of the first transversely UPO in the attractor A, we propose to call the bifurcation in which such UPO is created as the *generalized riddling bifurcation*.

If the attractor B does not exist, such a trajectory has to come back to the attractor A (trajectory γ_2 in Fig. 1(b)). In this case, the riddling is only local but the bursts off the attractor A allows it to grow in the direction which was not allowed before the appearance of O_1. We propose to call this phenomenon the *higher-dimensional bubbling*.

In the case of the riddling of higher-dimensional attractors the set $R = \{O_1, O_1^{-1}, O_1^{-2}, \ldots\}$ or $R = \{\overline{O}_1, \overline{O}_1^{-1}, \overline{O}_1^{-2}, \ldots\}$ is countably infinite. Due to the ergodicity, any trajectory γ on the attractor A has to visit the neighborhood of O_1 (or one of $O_1^{-1}, O_1^{-2}, \ldots, \overline{O}_1^{-1}, \overline{O}_1^{-2}, \ldots$) and, if perturbed off the attractor A, it leave the attractor in a finite time.

4. Example

As an example, consider a three-dimensional map F in the form:

$$
\begin{aligned}
x_{n+1} &= 1 + z_n - a y_n^2, \\
y_{n+1} &= 1 + b y_n - a x_n^2, \\
z_{n+1} &= b x_n,
\end{aligned}
\tag{2}
$$

where $x_n, y_n, z_n \in \mathbb{R}$ are dynamical variables, a and $b \neq 0$ are the system parameters. This map was introduced in [5] as an example of a simple system which shows both chaotic and hyperchaotic behaviour. Map (2) is invertible as its Jacobian is equal $-b^2$.

In our numerical simulations, we consider $b = 0.2$ and take a as a control parameter. For $a < 1.267$, map (2) has a chaotic attractor A with such a property that all UPO embedded in it have stable transverse directions. At $a = 1.2678$, the first UPO with unstable transverse direction appears initiating the sequence of bifurcations of UPOs which lead to the chaos–hyperchaos transition at $a = 1.297$. The example of the chaotic attractor A ($a = 1.26$) is shown in Fig. 2(a) and (b). The crosses in Fig. 2(a) and (b) indicate a period-2 UPO with one stable S_1 and one unstable U_1 directions along the attractor A and the stable direction S_2 transverse to A. By increasing the parameter a this period-2 UPO undergoes the bifurcation in which its transverse direction becomes unstable (S_2 becomes U_2). After the bifurcation one can observe tongues of points leaving the neighborhood of the attractor A in the transverse direction U_2, as shown in Fig. 2(c,d). These tongues anchor at O_1 and at the points $\overline{O}_1^{-1}, \overline{O}_1^{-2}, \ldots$ on the stable manifold S_1.

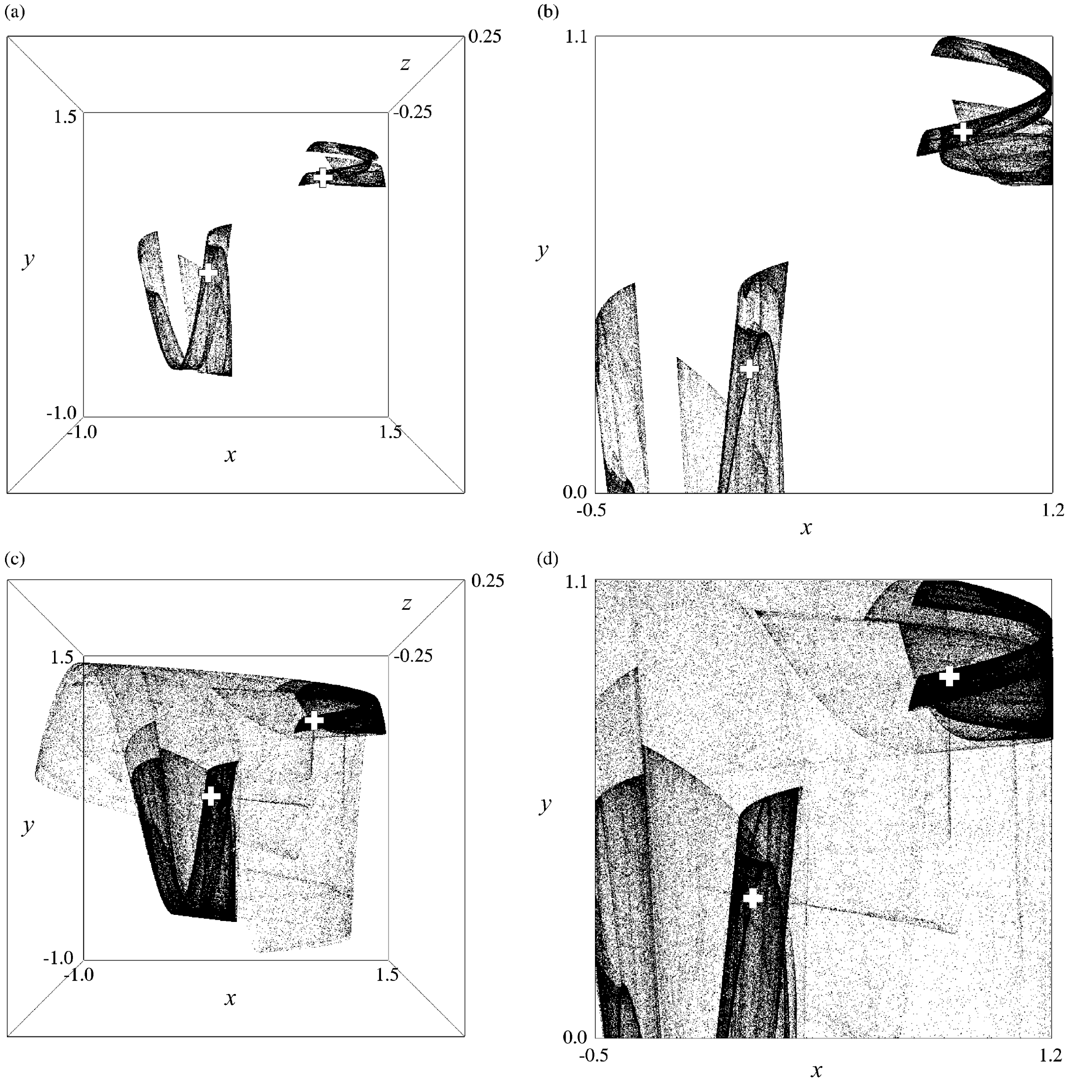

Fig. 2. Attractor of the map (2) for $b = 0.2$: (a) before the riddling bifurcation $a = 1.26$, (b) enlargement of (a), (c) after the riddling bifurcation $a = 1.27$, (d) enlargement of (c).

The basin of attractor A is riddled in the sense of definition introduced in Section 3. As A is the only attractor of the map (2) in the considered range of parameters, the trajectories leaving its neighborhood have to come back to it. We observe the phenomenon of the higher-dimensional bubbling.

5. Conclusions

In summary, we showed that the appearance of the first UPO with more than one unstable directions on the chaotic attractor allows the trajectory evolving in a neighborhood of the attractor to leave it along the direction transverse to the attractor. Trajectory which leaves the attractor, can be attracted by another attractor and this phenomena we described as generalized riddling. If the other attractor does not exist, such a trajectory comes back to the attractor and one has bubbling, i.e., the trajectory bursts in the direction transverse to the attractor along the invariant manifold. We believe that the bubbling of the higher-dimensional attractor is a typical way in which attractors grow.

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