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Influence of the mass and stiffness ratio on a periodic motion of two impacting oscillators

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Abstract

Dynamical behaviour of the system consisting of two impacting oscillators is the object of the investigations whose results are presented in this paper. The investigations were aimed at determination in which regions of the parameters characterising the system, the system motion is periodic. The investigations were carried out by preparing bifurcation diagrams and using a special method of analysis of analytical solutions of differential equations that describe the motion of this system.

1. Introduction

In the second half of the 1990s, a system consisting of two independent oscillators that could impact on each other during motion, became the object of the investigations carried out by the authors of this paper and their co-workers [2–5]. The motion of the upper oscillator is kinematically forced, whereas the lower oscillator motion results from impacts only (Fig. 1).

An example of the first results obtained by the authors has been shown in Fig. 2. This is a bifurcation diagram made for two identical oscillators ($\sigma = \mu = 1$) that touch each other in the static equilibrium position ($d = 0$). The motion of these oscillators is weakly damped with viscous damping that is equal to 0.001 of critical damping. In the bifurcation diagram, one can see two regions of the periodic motion (periods 1 and 2) that are important from the technical (application) viewpoint. Maximum amplitudes of periodic vibrations occur for $\eta = 2$ and 4, i.e., at the forcing frequencies equal to multiples of the frequency of free vibrations of the oscillators and, therefore, the result shown in Fig. 2 can be described as easy to be foreseen. During further investigations, it was found however that in the case of oscillators with various frequencies of free vibrations ($\mu \neq \sigma$), the ranges of the forcing frequency η in which the motion is periodic change in the way that is difficult to predict through analysis of the ratio of the frequency of free vibrations (which is the idea put forward by the authors and suggested by many other researchers as well).

The method described by Peterka [1,7,8] has turned out to be a suitable tool for seeking these regions of the parameters characterising the system for which the system motion is periodic. This method was used by Peterka to investigate of a system with one degree of freedom: an oscillator subjected to external forcing and impacting on a fixed base. An application of this method to a system consisting of two independent oscillators is presented by Czolczynski [6].

Below, the results of the investigations carried out both with the employment of bifurcation diagrams and with the method described by Peterka are discussed.

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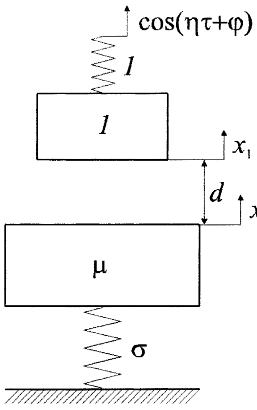

Fig. 1. Two impacting oscillators.

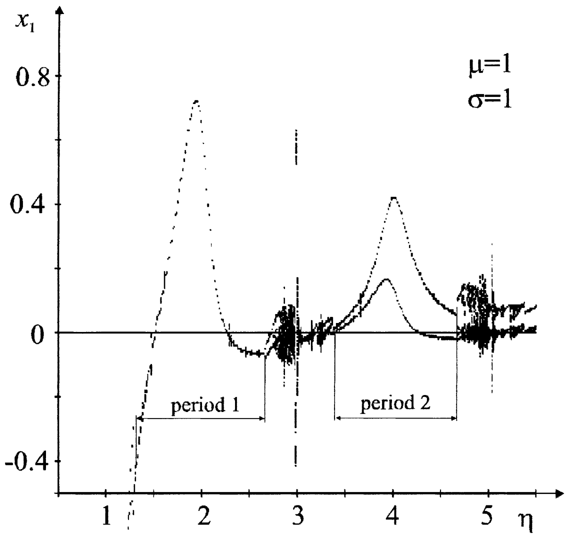

Fig. 2. Bifurcation diagram for $\mu = 1$ and $\sigma = 1$ with small damping.

An influence of the parameters σ and μ on the periodicity of the motion of oscillators has been illustrated by some examples. The existence of the phenomenon of the coexistence of periodic attractors, which implicates a necessity for studying the breadth of basins of attraction, has been pointed out to as well.

2. Mathematical model of the system

The mathematical model of the system presented in Fig. 1 is composed of equations describing impacts and two differential equations describing the motion of oscillators between subsequent impacts:

$$\begin{aligned} &\ddot{x}_1 + x_1 = \cos(\eta\tau + \varphi) \\ &\mu\ddot{x}_2 + \sigma x_2 = 0 \end{aligned} \tag{1}$$

The frequencies of free vibrations of both the oscillators are equal to: $\alpha_1 = 1$ and $\alpha_2 = (\sigma/\mu)^{1/2}$, respectively. Eq. (1) have analytical solutions of course, however taking into consideration a possibility of introducing non-linear terms into them, they were solved using the Runge–Kutta method to prepare bifurcation diagrams.

When the distance between the oscillators decreases to zero, that is to say, when:

$$x_1(\tau) + d = x_2(\tau) \tag{2}$$

an impact that is modelled with the following equations:

$$\begin{aligned} v_1^+ - v_1^- &= -\mu(v_2^+ - v_2^-) \\ v_1^+ - v_2^+ &= -k_r(v_1^- - v_2^-) \end{aligned} \tag{3}$$

takes place, where k_r denotes the coefficient of restitution, whereas v^- and v^+ stand for the velocity before and after impact, correspondingly.

As has been already mentioned Section 1, a method that consists in analysis of properties of analytical solutions to Eq. (1) was employed to seek the regions of the periodic motion. This method consists in determination of such sets of the parameters μ, σ, d, k_r and η for which the following conditions are satisfied:

1. Impacts take place every $n(=1,2,3,\ldots)$ periods of the external forcing $T = 2\pi/\eta$ and occur at the same displacements of the oscillators:

$$\begin{aligned} x_1(0) &= x_1\left(\frac{2\pi n}{\eta}\right) \\ x_2(0) &= x_2\left(\frac{2\pi n}{\eta}\right) \\ x_1(0) + d &= x_2(0) \end{aligned} \tag{4}$$

2. At the instant of impact, Eq. (3) are fulfilled.
3. Between subsequent impacts, the solutions do not cross each other, i.e., the oscillators do not penetrate each other, which, of course, would be impossible from the physical point of view:

$$x_1(\tau) + d > x_2(\tau) \text{ for } 0 < \tau < \frac{2\pi n}{\eta} \tag{5}$$

4. The solutions are stable.

A detailed description of this method is to be found in [6]. As the application of this method by Czolczynski has been inspired by the papers written by Peterka (and by personal communication with F. Peterka), the method will be referred to as Peterka's method henceforward.

3. Influence of the mass ratio on the existence of a periodic motion

Fig. 3 shows a bifurcation diagram made for the system whose parameters have the following values: $\mu = 4$, $\sigma = 1$, $d = 0$, $k_r = 0.6$. The displacements of the upper oscillator $x_1(\tau)$ at the instants distant by one period T of the external forcing, for various values of the frequency η are depicted in this diagram. During the computations, after each change in the value of η, the time equal to 800 T was neglected (as a transient period). One can see in this diagram that the ranges of the frequency η in which the motion is periodic are narrower than in Fig. 1, and, moreover, their midpoints fall at $\eta \approx 1.5$ (period 1), $\eta \approx 3.2$ (period 2) and $\eta = 4.9$ (period 3) instead of $\eta = 2$, $\eta = 4$ and $\eta = 6$, as was the case for $\sigma = \mu = 1$. Fig. 4 presents time series with examples of a periodic motion of the system, for $\eta = 1.3$ (period 1–4a), $\eta = 3.25$ (period 2–4b) and $\eta = 4.9$ (period 3–4c). As can be seen, in all these examples impacts occur at the same displacements and velocities (are identical), that means, that the first condition from the Peterka method is fulfilled.

For the frequencies of the external forcing that are outside of the "periodic" ranges indicated in Fig. 3, the system moves with a chaotic motion whose example has been shown in Fig. 5(a) (time series) and Fig. 5(b) (Poincare map).

Fig. 6(a) shows a result of seeking for the regions of the existense of a periodic motion, obtained by means of Peterka's method, for the system with the following parameters $\sigma = 1$, $d = 0$, $k_r = 0.6$ and the parameter μ varying from $\mu = 0.5$ to 10.5 . In the regions of the parameters η and μ indicated by the gray colours, the system moves with the periodic motion with period 1, period 2, or period 3. The changes in the position and width of the frequency η ranges, caused by an increase in the mass of the lower oscillator for which the motion is periodic, are shown in this figure.

As one can see, these variations are highest for small values of μ, that is to say, where this parameter has the strongest influence on the frequency of free vibrations α_2 and the amplitude of vibrations of the lower oscillator: see

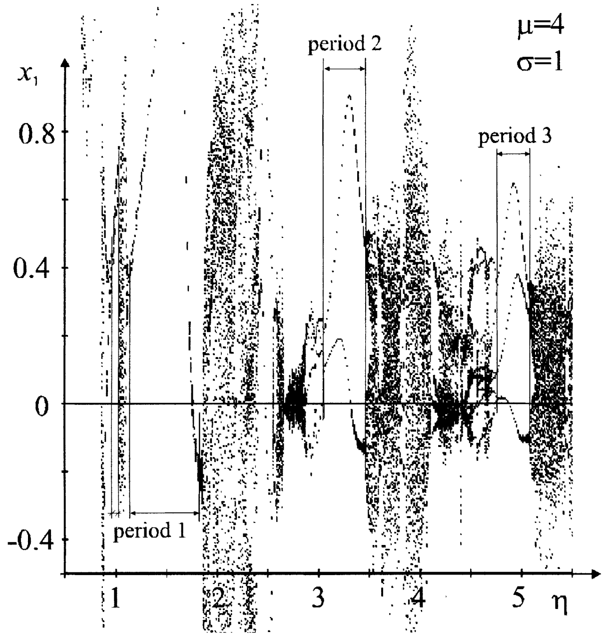

Fig. 3. Bifurcation diagram for $\mu = 4$ and $\sigma = 1$.

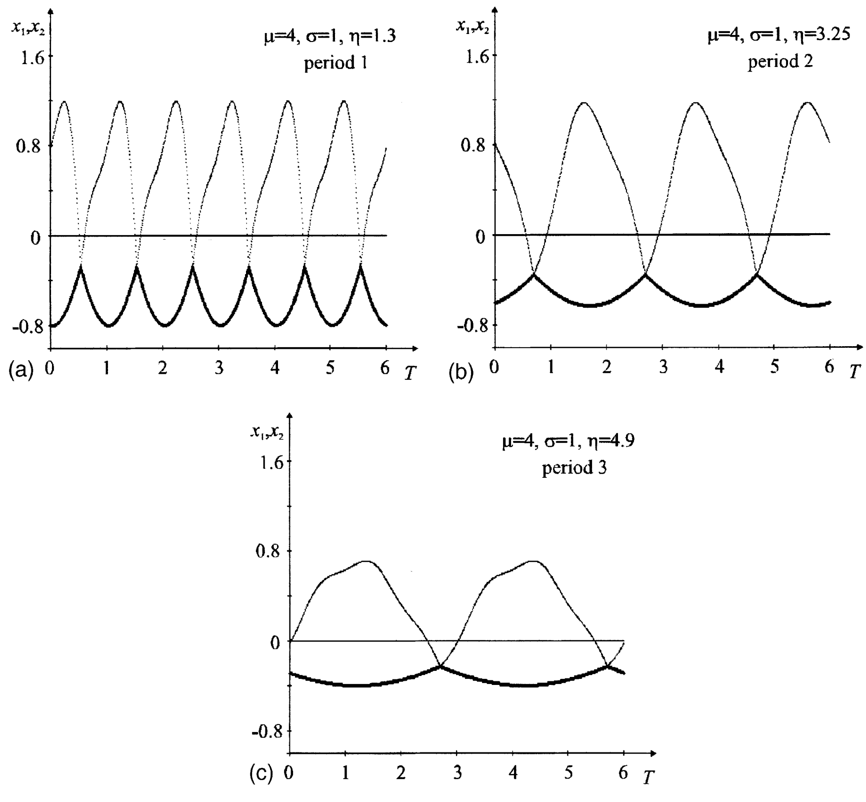

Fig. 4. Examples of periodic motion of two impacting oscillators.

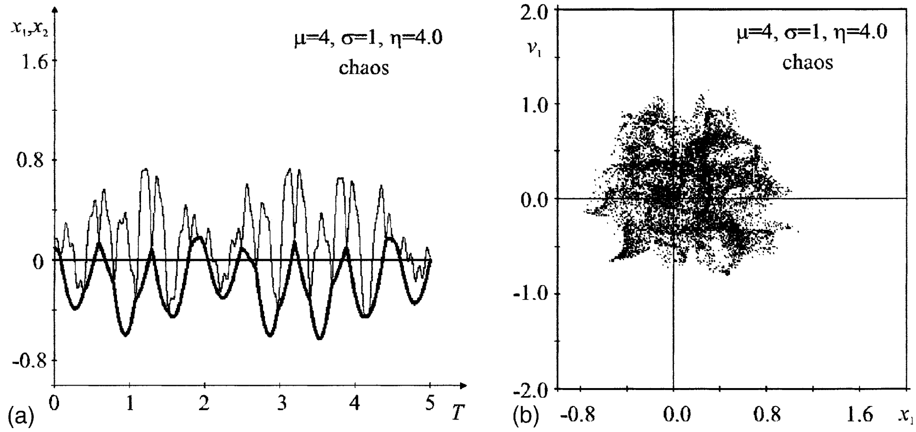

Fig. 5. Chaotic motion ((a) time diagram, (b) Poincare map) of two impacting oscillators.

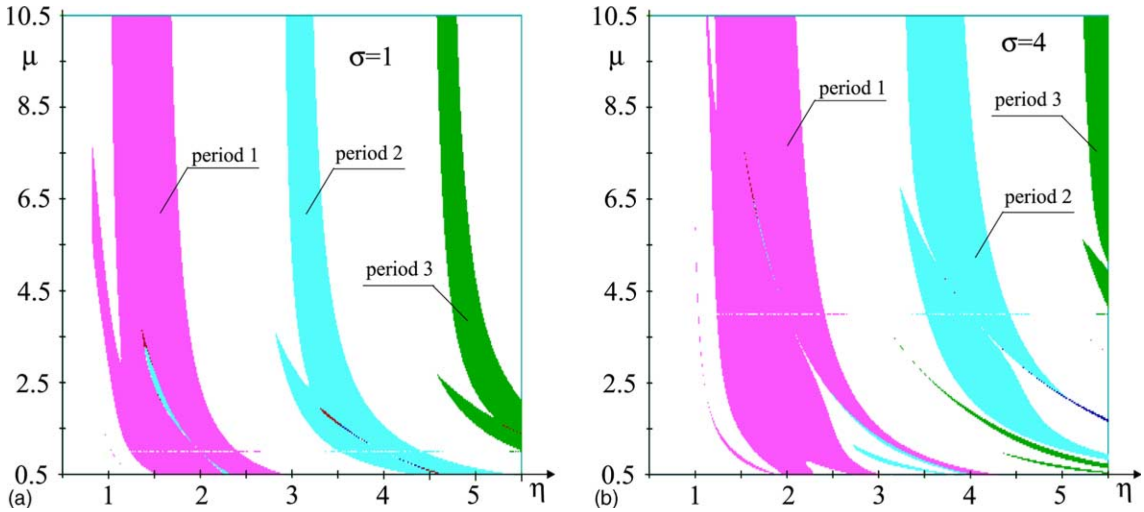

Fig. 6. Regions of existence of a periodic motion obtained from Peterka's method for $\sigma = 1$ (a), $\sigma = 4$ (b) and various values of μ.

time series for $\mu = 1.5$ and $\eta = 1.5$ in Fig. 7(a). For high values of μ, the amplitude of vibrations of the lower oscillator is so small that its changes (and a change in α_2) do not affect significantly the motion of the upper oscillator: compare Fig. 7(b) ($\eta = 1.5$, $\mu = 5$) and Fig. 7(c) ($\eta = 1.5$, $\mu = 10$), which are very similar to each other. Therefore, changes in the ranges of η are very slight for high values of μ.

In Fig. 6(a) non-continuity of the regions of the periodic motion, observed for $\mu = 1$, is to be noticed. As turns out, in the case of $\mu = \sigma = 1$ (and in all other cases of $\sigma = \mu$), the system motion is never asymptotically stable: the satisfactory condition for stability according to the Hurwitz criterion is not fulfilled. For these values of η, for which the system motion was periodic under week damping (see Fig. 2), a quasi-periodic motion is observed now—see a bifurcation diagram in Fig. 8(a). In this system, a phenomenon of torus doubling that leads from a quasi-periodic motion to chaos has been observed as well. Poincare maps, which illustrate this process, are shown in Fig. 9.

On the map that is shown in Fig. 6(a), the regions of the periodic motion that exist in the subresonance range ($\eta < 1$) and that are characterised by numerous and various (as far as the position and velocity are concerned) impacts occurring per one period of the external forcing have not been marked. An example of such a motion has been depicted in Fig. 10—$\eta = 0.63$—period 3. The regions of the existence of such a kind of the periodic motion cannot, however, be found with Peterka's method in its present form, as the condition 1 is not fulfilled.

In Fig. 6(b) one can see a map analogous to that one in Fig. 6(a) but made for the parameter $\sigma = 4$. An increase in the stiffness of the lower oscillator spring has been followed by an increase in the width of the regions of the periodic

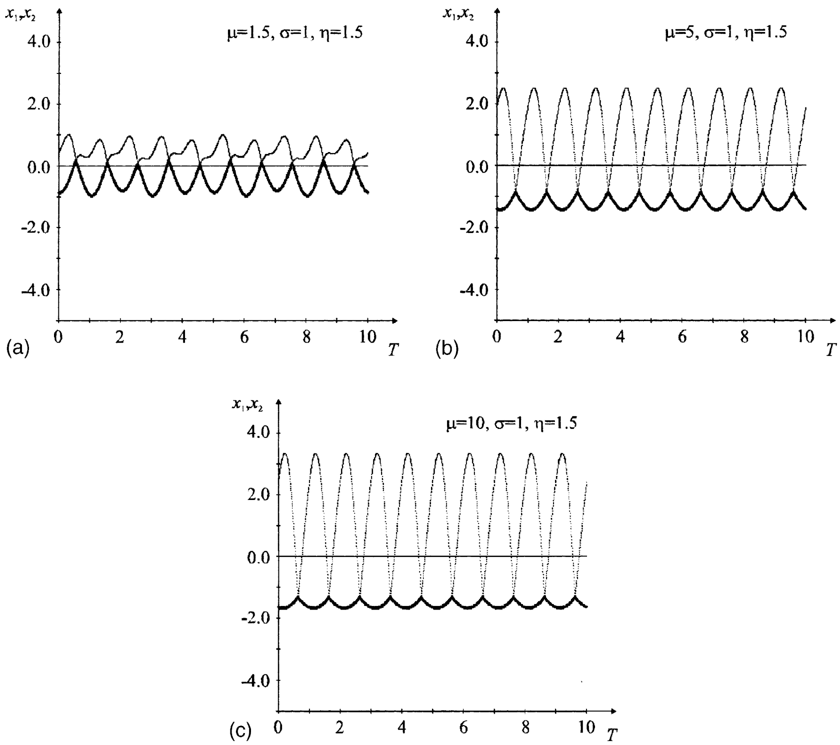

Fig. 7. Periodic motion of the system with various ratios of masses: (a) $\mu = 1.5$, (b) $\mu = 5$, (c) $\mu = 10$.

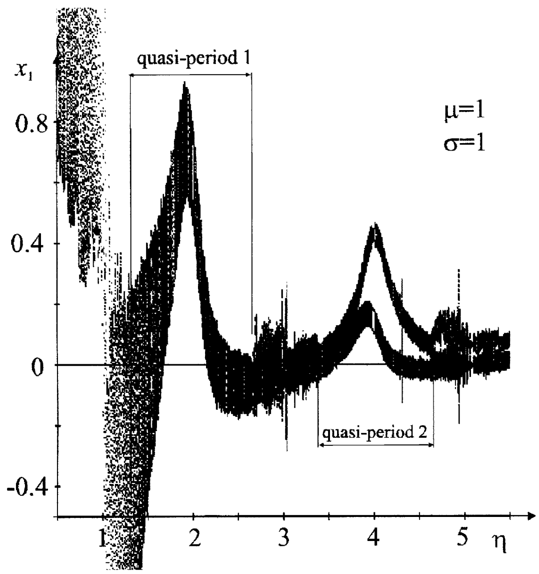

Fig. 8. Bifurcation diagram for $\mu = 1$ and $\sigma = 1$ without damping.

motion and their displacement towards higher values of the forcing frequency. For $\mu = 4$ $(= \sigma)$, non-continuity of the regions caused by a change from a stable periodic motion into a quasi-periodic one can be observed.

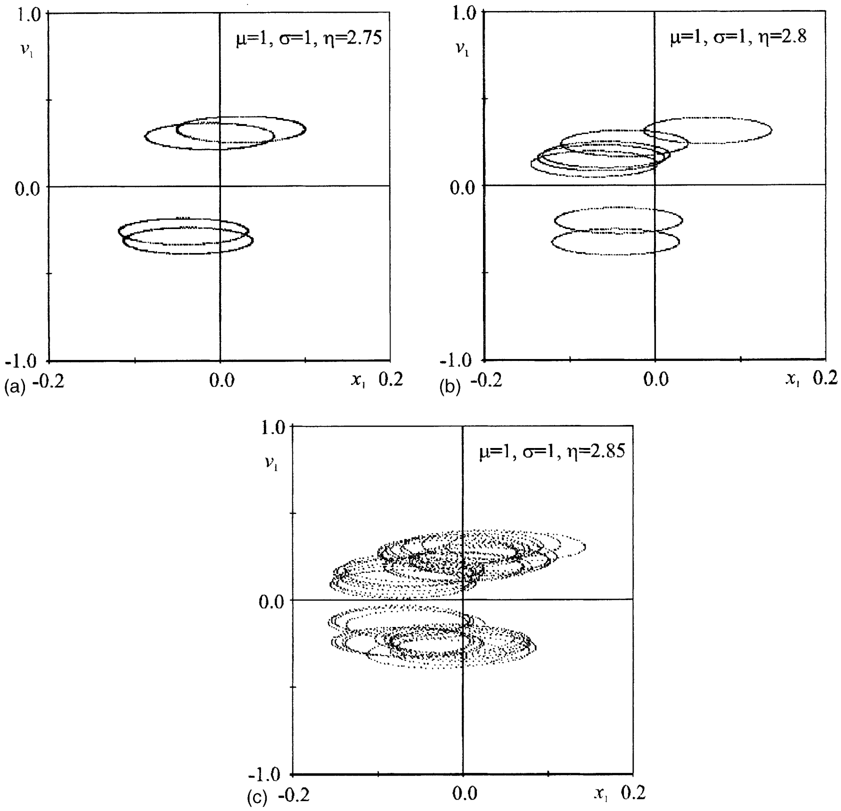

Fig. 9. Torus doubling—a road from quasi-periodic motion to chaos.

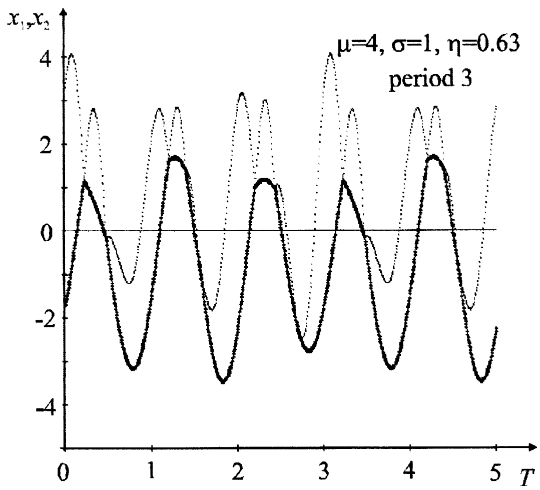

Fig. 10. Periodic motion of the system with various impacts.

The most significant changes in the shape of the regions of the periodic motion are observed for $\mu \ll \sigma$, i.e., where the frequency of free vibrations of the lower oscillator is higher than the frequency of free vibrations of the upper oscillator.

4. Influence of the stiffness ratio on the existence of a periodic motion

Fig. 11(a) shows an analogous map of the regions of the periodic motion to that one in Fig. 6, but made for a constant value of the mass ratio $\mu = 2$ and the stiffness ratio σ varying in the range from 0 to 10 ($d = 0, k_r = 0.6$). In

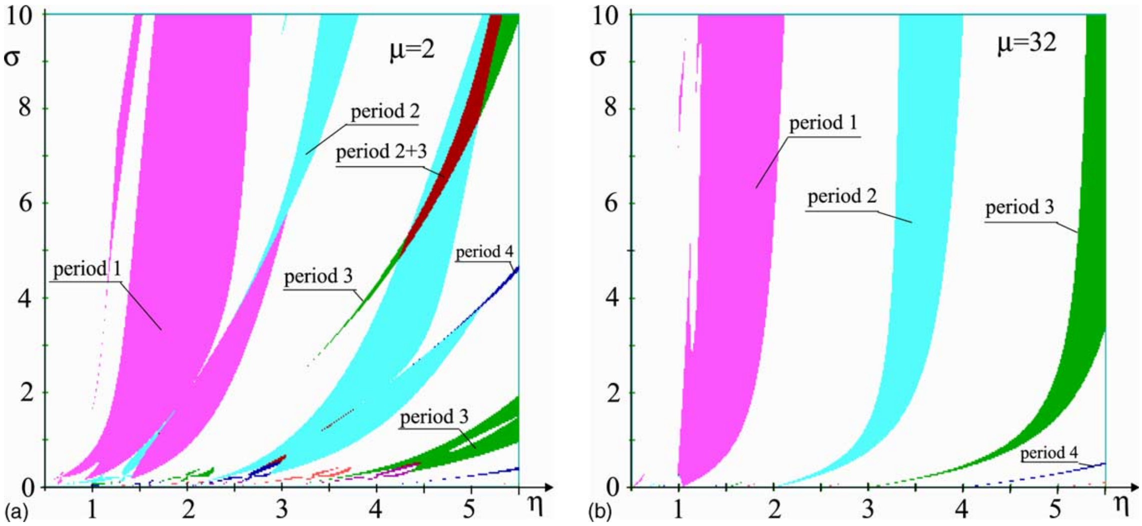

Fig. 11. Regions of existence of a periodic motion obtained from Peterka's method for $\mu = 2$ (a), $\mu = 32$ (b) and various values of σ.

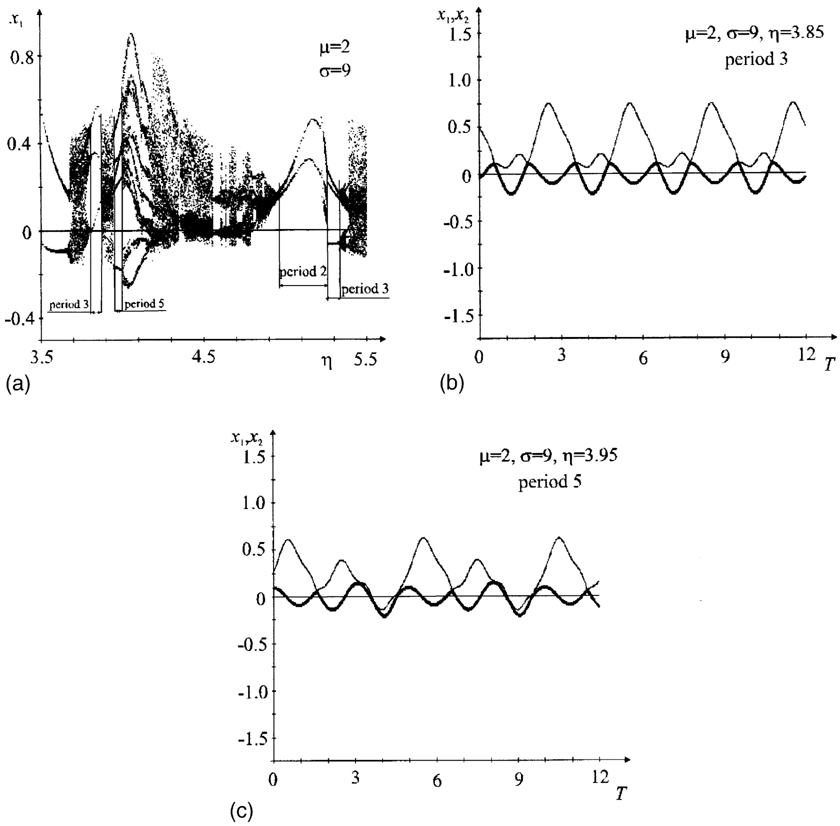

Fig. 12. Bifurcation diagram for $\mu = 2$ and $\sigma = 9$ (a) and examples of periodic motion with various impacts.

Fig. 11(a) attention should be paid to the region of the coexistence of two various kinds of the periodic motion—periods 2 and 3. In the case of the coexistence of two various kinds of motion, a question arises on the size of the basins of attraction of both the attractors, i.e., a question on the sensitivity of motion to disturbances.

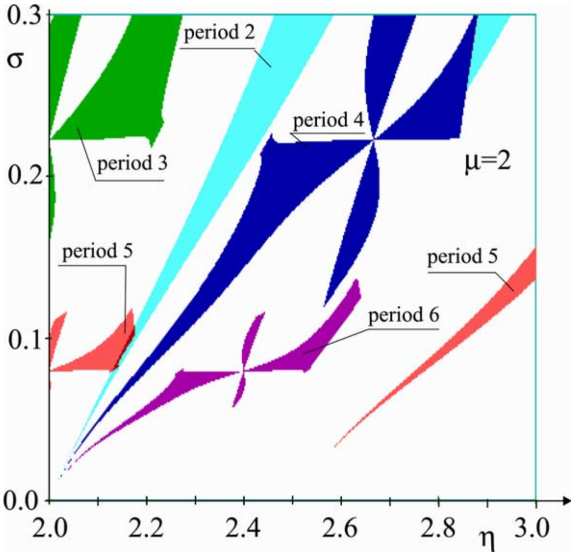

Fig. 13. Regions of existence of a periodic motion obtained from Peterka's method for $\mu = 2$ in the case of small values of σ.

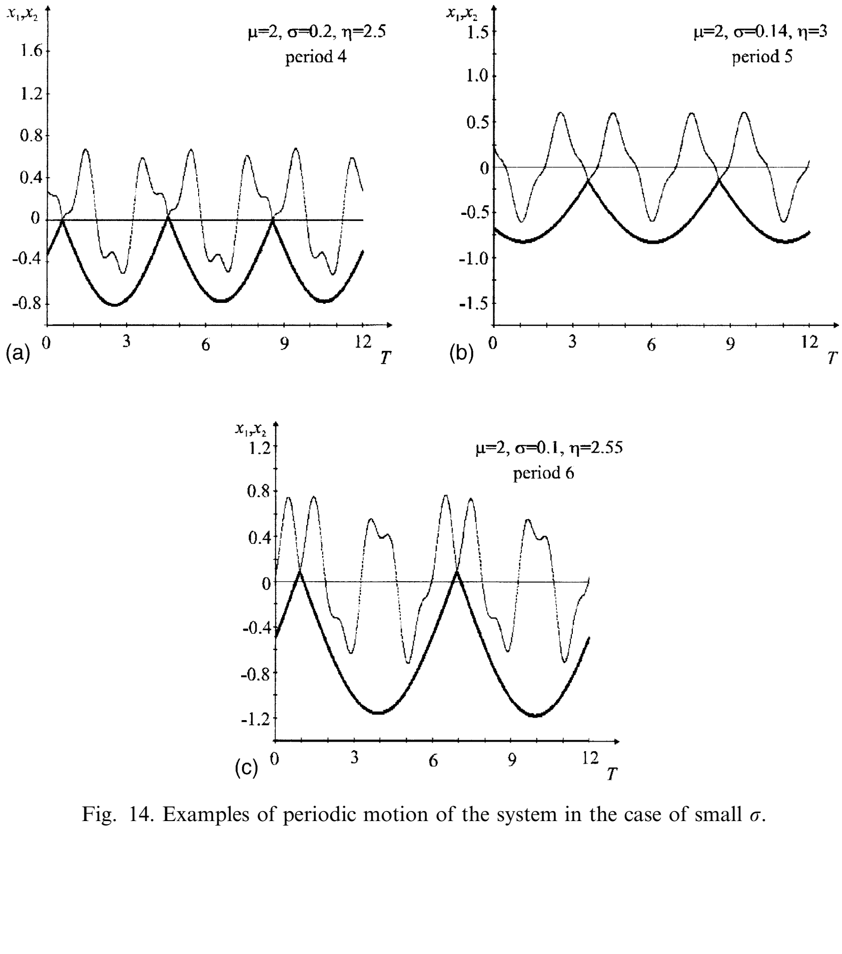

Fig. 14. Examples of periodic motion of the system in the case of small σ.

Fig. 12(a) presents a bifurcation diagram made for $\sigma = 9$, in the range $3.5 \leqslant \eta \leqslant 5.5$. Apart from wide ranges of the periodic motion (periods 2 and 3), which can also be observed in Fig. 11, one can see narrow ranges of the periodic motion (for instance, period 3 for $\eta = 3.85$, period 5 for $\eta = 3.95$) which are not present in Fig. 11. The periodic motion existing in these ranges is characterised by numerous different impacts occurring per one period of the external forcing and cannot be identified by Peterka's method. Some examples of time diagrams of such periodic motions have been depicted in Fig. 12(b) and (c).

Similarly as in Fig. 6, non-continuity of the regions of the periodic motion, caused by a change from the periodic motion to a quasi-periodic one, can be seen in Fig. 11(a) for $\sigma = 2$ $(= \mu)$.

For low values of σ, which means a low value of the frequency of free vibrations of the lower oscillator, small regions of the periodic motion with periods 4, 5, 6 can be observed in Fig. 11(a). They are depicted in Fig. 13. Exemplary time series are presented in Fig. 14(a)–(c) for periods 4, 5, 6, correspondingly.

Fig. 11(a) shows regions of the periodic motion of the systems in which (for $\sigma > \mu = 2$) the frequency of free vibrations α_2 of the lower oscillator is higher than the frequency of free vibrations of the upper oscillator. This $(\alpha_2 > \alpha_1)$ explains the complexity of the shape of the periodic motion regions.

On the map shown in Fig. 11(b), for $\mu = 32$, $\sigma = 0$–10, the frequency of free vibrations of the lower oscillator α_2 is lower than the frequency of free vibrations of the upper oscillator, and like in Fig. 6b, higher regularity of the shape of the periodic motion regions can be seen.

5. Conclusions

The conducted numerical investigations have confirmed the usefulness of Peterka's method for determination of the regions of the existence of a periodic motion of two impacting oscillators with various masses and stiffness coefficients of springs.

An agreement between the results obtained by means of Peterka's method and commonly used bifurcation diagrams has been found.

It is impossible to identify the regions of the existence of a periodic motion during which impacts that differ as far as their position and velocity are concerned, i.e., which are not identical, with Peterka's method (in its present form). On the other hand, owing to the fact that the regions of the existence of such solutions (in terms of the width of the range of η) observed in the bifurcation diagrams are narrow, the usefulness of such solutions from the engineer's point of view seems to be insignificant.

The investigations whose results have been discussed in the present paper concern a system for which $d = 0$ (the oscillators touch each other in the static equilibrium position). In systems in which d is higher than the amplitude of forced vibrations of the main oscillator, the coexistence of the motion with and without impacts always occurs, which restricts applications of these systems.

In his future research, the authors will investigate an influence of other parameters of the system, such as the coefficient of restitution and the damping coefficient on the existence of a periodic solution to the equations of motion.

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