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ENERGY METHOD FOR POSITION STABILITY ANALYSIS OF CRITICAL POINTS OF ROBOT MANIPULATORS

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An energy method for the position stability analysis of critical points (equilibrium positions) of dynamical systems with an arbitrary finite number of degrees of freedom is presented. The method proposed consists in the analysis of the potential energy balance of a perturbation of an arbitrary number of generalised co-ordinates of the dynamical system in order to establish the energy stability of individual degrees of freedom and of the whole system. In the case of mechanical systems, the position stability criterion, understood as minimisation of loads of driving systems with conservative forces, is discussed. Some examples of application of the method to the position stability analysis of critical points of one- and two-degree-of-freedom systems are analysed.

Key words: critical point, load of driving system, position stability

1. Introduction

A physical system in the equilibrium position is subject to various types of external perturbations. Position perturbations can be caused both by external loads acting on the system and by inaccuracies of the physical system positioning. The problem of near-critical analysis of a dynamical system subject to a perturbation of generalised co-ordinates has practical meaning for the determination of a tendency of the system behaviour. The dynamical system remaining in the vicinity of the equilibrium position, such as the critical point, is stable in the positioning sense if there is a tendency towards conservation of the equilibrium position despite the effect of perturbations. Otherwise, the

critical point is unstable. Additionally, in the case of mechanical systems, it is advantageous to position links in such a way so that driving systems are unloaded. Critical points, which are equilibrium positions, are such special configurations of mechanical system links, in which the work necessary to maintain the system at rest is minimised, and thus the driving system loading with conservative forces is decreased.

In general, among methods of stability analysis of dynamical systems, we can distinguish methods based on analysis of equations of motion of a system and energy methods. Commonly used methods of the stability analysis of critical points are methods based on eigenvalues (Kapitaniak, 2001; Szumiński, 1995; Wiggins, 1990) or Lapunov's exponents (Kapitaniak, 1991; Kapitaniak and Wojewoda, 2001; Ott *et al.*, 1990; Wiggins, 1990). These methods allow for a qualitative description of the dynamical system behaviour in the vicinity of the critical point through the statement of its stability or instability. Well-known are also methods related to limitations of solutions to equations of motion, for instance investigations of the stability of solutions to equations of motion in terms of Lagrange's or Lapunov's analysis (Kapitaniak, 1991; Gutowski and Świetlicki, 1986; Demidowicz, 1972). In turn, a method based on perturbation equations (Szumiński, 1996, Minorsky, 1962) that consists in analysis of equations of a perturbed motion allows for analysis of the dynamical system behaviour in the vicinity of the critical point as a function of one or more motion perturbations introduced to the system. Among the energy methods, well known is a method of analysis of derivatives of the potential energy of a dynamical system in the critical point. It is possible to obtain an answer concerning the stability of the dynamical system in the vicinity of the critical point in an easy way on the basis of analysis of the sign of the second derivative of the system potential energy with respect to the generalised co-ordinate for parameters of the critical point (Kapitaniak, 2001; Cesari, 1959; Langhaar, 1962; Kruszewski *et al.*, 1993; Schuster, 1993). Employing the principle of virtual works, one can show that the necessary and sufficient conditions for the stability of critical points can also be generalised for systems with many degrees of freedom. A separate group of methods for the investigations of stability of some defined types of dynamical systems are stability criteria, for example Routh-Hurwitz's criterion, which are used for the stability analysis of dynamical systems described by linear equations of motion with constant coefficients (Kheir, 1996).

The method proposed here is an energy method. It allows one to answer the question concerning the position and energy stability of the critical point, and to carry out quantitative analyses of the dynamical system behaviour in

the vicinity of the critical point as a function of the number and magnitude of the perturbation of generalised co-ordinates, without necessity of formulation of differential equations of motion. The method rests on recording the perturbation potential energy of a dynamical system and on the static analysis of its changes in the vicinity of the critical point. It allows one to determine the character of motion in the vicinity of the critical point.

The paper consists of four sections. Theoretical grounds of the method are discussed in the second part. The third section presents examples of the stability analysis and determination of characteristics of critical points of exemplary dynamical systems. Dynamical systems with one and two degrees of freedom have been assumed as examples. In the fourth part conclusions are included.

2. Theoretical grounds

Let us assume that a vector of generalised co-ordinates of a dynamical system assumes the following form

$$\overline{q} = [q_1, q_2, ..., q_i, ..., q_n]^\top \tag{2.1}$$

where $q_i \in \overline{M}$ and M is the set of perturbed generalised co-ordinates of the dynamical system.

Let us now introduce a perturbation of the ith generalised co-ordinate in the form

$$q_i = q_i + \varphi_i \tag{2.2}$$

where φ_i is the perturbation of the ith generalised co-ordinate of the system.

Let us assume that the relation expressing the potential energy of the nominal system position as a function of its vector of the generalised co-ordinates is known. The potential energy, taking into account the effect of a perturbation of selected generalised co-ordinates, is expressed as follows

$$E_p = A + Z(\varphi_i) \tag{2.3}$$

where A are potential energy terms dependent on the location of the nominal system in the system state space and the data concerning the mass and geometry of the system structure. Here, the potential energy terms describing the nominal position of the dynamical system are included; $Z(\varphi_i)$ are terms of the potential energy equation describing perturbations of the dynamical system position.

In general, the potential energy of the dynamical system is expressed as

$$E_p = E_p^n + E_p^p \tag{2.4}$$

where E_p^n is the potential energy of the nominal position of the dynamical system, and E_p^p – potential energy of the nominal position perturbation.

As a result of the assumption of the analysis in the close vicinity of the dynamical system equilibrium points, the trigonometric functions describing the potential energy are replaced by a Taylor series. The obtained relations represent changes of the potential energy of the perturbation in vicinities of individual configurations in the system state space as a function of the perturbation magnitude of individual generalised co-ordinates and critical point parameters in the space of motion parameters of the dynamical system.

The perturbation potential energy is understood as the work that should be performed to bring the perturbed dynamical system to the equilibrium position.

In order to carry out an analysis of the system dynamics in neighbourhoods of critical points, the stability criteria based on the following cases are given.

The dynamical system is stable in the position sense in the critical point if each generalised co-ordinate is stable in this point, that is to say, when

$$\forall i \quad E_{pi}^p > 0 \qquad i = 1, 2, ..., n$$

The dynamical system is unstable in the critical point if at least one generalised co-ordinate of the dynamical system is unstable, i.e. when

$$\exists i \quad E_{pi}^p < 0 \qquad i = 1, 2, ..., n$$

We can distinguish the following cases of the sign of the perturbation potential energy of the dynamical system:

- for $E_p^p < 0$ a loss of the system potential energy resulting form the occurrence of perturbations; energy should be delivered to the system (work should be performed), the equilibrium point is unstable – some generalised co-ordinates can be stable; when upset from the equilibrium position, the system escapes from the critical point;

- for $E_p^p > 0$ an increase in the system potential energy resulting from the occurrence of perturbations; energy can be taken from the system as a whole, the equilibrium point can be stable or unstable, depending on the signs of values of potential energies of the perturbation E_{pi}^p of individual generalised co-ordinates of the system; in the case of the stable system, when upset from the equilibrium position, it tends to return to it;

- for $E_p^p = 0$ lack of perturbations, i.e. the perturbation potential energy equals to zero; in the case when perturbations take place, a stability boundary of the dynamical system in the energy sense occurs.

We can distinguish the following cases of the sign of the perturbation potential energy of the ith generalised co-ordinate of the dynamical system:

- for $E_{pi}^p < 0$ when the position is upset, energy should be supplied to the system (work should be performed) to bring the system to the equilibrium position; the generalised co-ordinate is unstable; in mechanical systems this case is disadvantageous due to an increase in the load of the driving system of the generalised co-ordinate;

- for $E_{pi}^p > 0$ after the position is upset, the system returns to the equilibrium position; energy can be taken back from the system; the generalised co-ordinate is stable; in mechanical systems this case is advantageous due to the fact that the driving system of the generalised co-ordinate self-unloads.

Apart from the presented position stability criterion of the vicinity of the critical point, an energy criterion can be assumed. It is as follows: a dynamical system is stable in the energy sense when the balance of the energy delivered to the system and taken back from it (i.e. its individual generalised co-ordinates) in order to bring the system to the critical point is positive, that is to say, when the potential energy of the system perturbations is positive. This criterion has been assumed in the analysis of the energy necessary to maintain the system in the critical point, and to eliminate stresses related to the position perturbation in driving systems.

As a result of numerical simulations, one can perform an analysis of changes in the potential energy with respect to the perturbation location of subsequent generalised co-ordinates of a dynamical system as a function of the perturbation magnitude, and for the case when:

- perturbations of a selected generalised co-ordinate of the dynamical system under consideration occur;

- perturbations of a higher number or of all generalised co-ordinates of the system under consideration occur.

As a result, some information on types and characters of individual critical points of the dynamical system analysed can be, for instance, obtained.

In the next section, examples of the application of the method to the analysis of types of characteristic points of sample dynamical systems is presented.

3. Analysis of characteristic points of a double pendulum

In the presented example only gravitational forces occur. The potential energy of the double pendulum in Fig. 1 with respect to the X-axis is

$$E_p = \frac{1}{2}g[m_1L_1 + m_2(2L_1 + L_2) - L_1(m_1 + 2m_2)\cos q_1 - m_2L_2\cos q_2] \quad (3.1)$$

where g is the gravity acceleration.

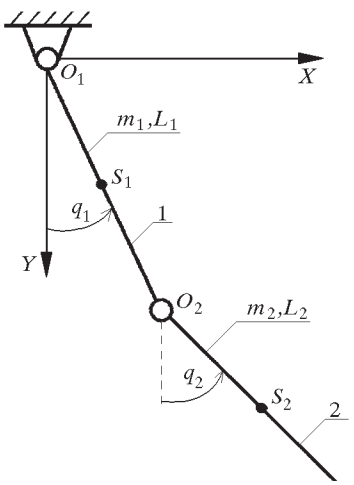

Fig. 1. Double pendulum

Now we introduce a perturbation of each generalised co-ordinate, Eq. (2.2)

$$q_1 = q_1 + \varphi_1 \qquad q_2 = q_2 + \varphi_2 \quad (3.2)$$

Introducing Eq. (3.2) into Eq. (3.1), the potential energy is

$$E_p = A + B\cos(q_1 + \varphi_1) + C\cos(q_2 + \varphi_2) \quad (3.3)$$

where the terms of the potential energy describing the basic motion are

$$A = \frac{1}{2}g[m_1L_1 + m_2(2L_1 + L_2)]$$

and the terms of the potential energy describing the perturbation are as follows

$$B = -\frac{1}{2}L_1(m_1 + 2m_2)g \qquad C = -\frac{1}{2}m_2L_2g$$

The potential energy of the system connected with the perturbations, Eq. (2.4), is equal to

$$E_p^p = B\cos(q_1+\varphi_1) + C\cos(q_2+\varphi_2) \tag{3.4}$$

Now the trigonometric functions in Eq. (3.4) are replaced by a Taylor series, and the series terms higher than four are rejected. Then, the potential energy of the system perturbations is equal to

$$\begin{aligned} E_p^p &= -B\Big[\Big(\frac{\varphi_1^2}{2}-\frac{\varphi_1^4}{24}\Big)\cos q_1 + \Big(\varphi_1 - \frac{\varphi_1^3}{6}+\frac{\varphi_1^5}{120}\Big)\sin q_1\Big] + \\ &- C\Big[\Big(\frac{\varphi_2^2}{2}-\frac{\varphi_2^4}{24}\Big)\cos q_2 + \Big(\varphi_2 - \frac{\varphi_2^3}{6}+\frac{\varphi_2^5}{120}\Big)\sin q_2\Big] \end{aligned} \tag{3.5}$$

If we assume the following values of the system parameters

$$m_1 = 1\,\mathrm{kg} \qquad m_2 = 1\,\mathrm{kg} \qquad L_1 = 0.2\,\mathrm{m} \qquad L_2 = 0.2\,\mathrm{m}$$

the coefficients B and C are equal to

$$B = -2.943\,\mathrm{Nm} \qquad C = -0.981\,\mathrm{Nm}$$

Taking into account the parameters of critical points of the system, the potential energy of perturbations, Eq. (3.5), assumes forms presented in Table 1.

Table 1

Parameters of the system in the critical point	Potential energy of perturbations of the system
$q_1 = 0,\ q_2 = 0$	$E_p^p = -B\Big(\frac{\varphi_1^2}{2}-\frac{\varphi_1^4}{24}\Big) - C\Big(\frac{\varphi_2^2}{2}-\frac{\varphi_2^4}{24}\Big)$
$q_1 = 0,\ q_2 = \pi$	$E_p^p = -B\Big(\frac{\varphi_1^2}{2}-\frac{\varphi_1^4}{24}\Big) + C\Big(\frac{\varphi_2^2}{2}-\frac{\varphi_2^4}{24}\Big)$
$q_1 = \pi,\ q_2 = 0$	$E_p^p = B\Big(\frac{\varphi_1^2}{2}-\frac{\varphi_1^4}{24}\Big) - C\Big(\frac{\varphi_2^2}{2}-\frac{\varphi_2^4}{24}\Big)$
$q_1 = \pi,\ q_2 = \pi$	$E_p^p = B\Big(\frac{\varphi_1^2}{2}-\frac{\varphi_1^4}{24}\Big) + C\Big(\frac{\varphi_2^2}{2}-\frac{\varphi_2^4}{24}\Big)$

The expressions in Table 1 allow for an analysis of changes of the potential perturbation energy in the close vicinity of equilibrium positions as a function of the perturbation magnitude φ_1 and φ_2.

3.1. Examples of numerical analysis

In Fig. 2, cases of stability and instability of the system in the vicinity of individual critical points for a perturbation of the generalised co-ordinate q_2 are presented.

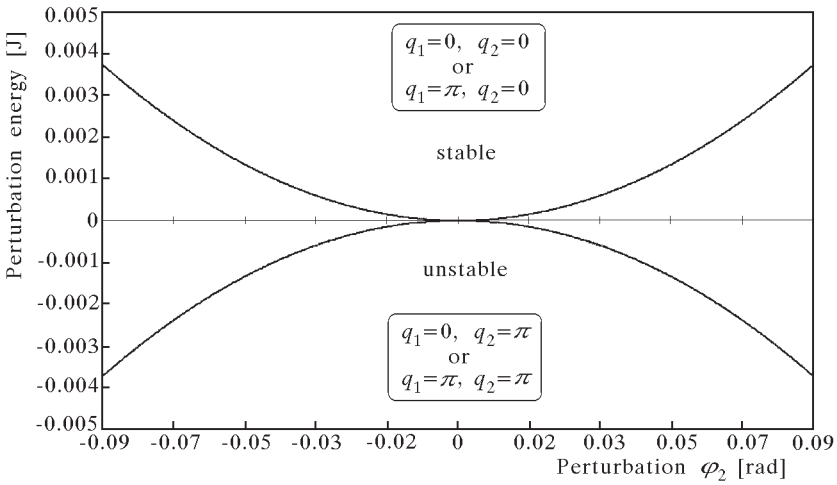

Fig. 2. Perturbation potential energy distribution in the case of an unstable and stable point

The sign of the perturbation energy decides about stability of the system in the vicinity of the equilibrium position, according to the assumptions presented in Section 2. A perturbation of one generalised co-ordinate fulfils the basic criterion of stability.

Let us consider an influence of both the generalised co-ordinates on the system stability. In Fig. 3, the vicinity of the equilibrium position: $q_1 = 0$, $q_2 = \pi$ is shown.

The potential energy of the system perturbation has a positive value, $E_p^p > 0$. The system satisfies the energy criterion of stability. However, the system is unstable as the generalised co-ordinate q_2 is unstable, $E_{p2}^p < 0$.

On the other hand, the potential energy of the system perturbation in the vicinity of the critical point $q_1 = q_2 = 0$ is presented in Fig. 4. The system is stable near the critical point because both the generalised co-ordinates of the system are stable, E_{p1}^p, $E_{p2}^p > 0$, that is to say, the potential energy of the system perturbation is positive, $E_p^p > 0$. The system fulfils also the energy criterion of stability.

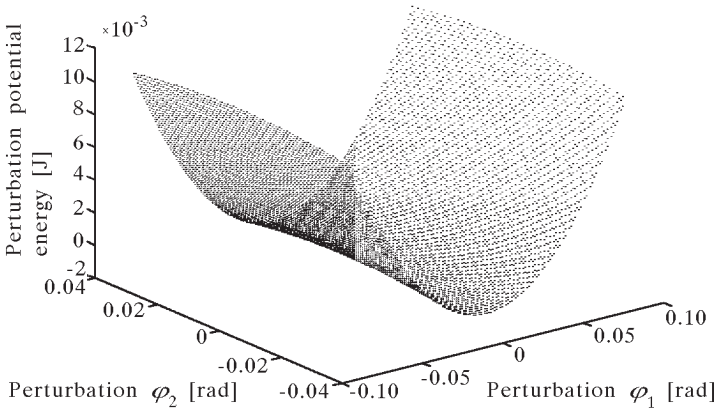

Fig. 3. Energy stability of the equilibrium $q_1 = 0$, $q_2 = \pi$

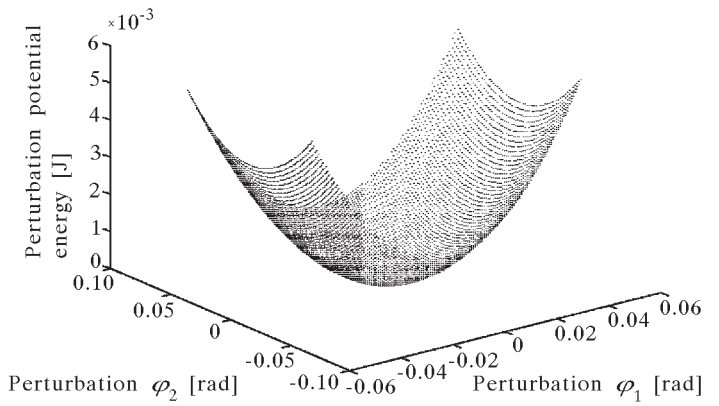

Fig. 4. Stability of the equilibrium $q_1 = 0$, $q_2 = 0$

In Fig. 5 the potential energy of the system perturbation in the vicinity of the critical point $q_1 = \pi$, $q_2 = 0$ is shown. In this case, the generalised co-ordinate q_1 is unstable, $E^p_{p1} < 0$, whereas the co-ordinate q_2 is stable, $E^p_{p2} > 0$. The system under consideration is unstable, which also follows from the fact that the value of the potential energy of the system perturbation is negative, $E^p_p < 0$. In this case, the system is not stable in terms of energy.

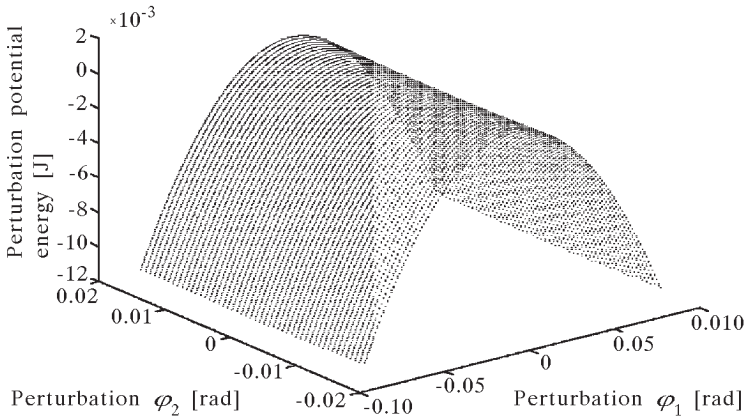

Fig. 5. Unstability of the equilibrium $q_1 = \pi$, $q_2 = 0$

4. Pendulum with an elastic support

Let us consider a system presented in Fig. 6. The potential energy of conservative forces in the form of forces of gravity and forces of elasticity, expressed in a system of the co-ordinates XY, is as follows

$$E_p = \frac{1}{2}kq^2 - \frac{1}{2}mgL(1 - \cos q) \tag{4.1}$$

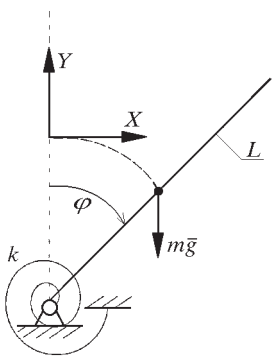

Fig. 6. Elastically suported pendulum

As we know, equating he potential energy derivative, Eq. (4.1), with respect to the variable q to zero, we obtain the equilibrium points $q = 0$ and a solution in the form

$$2kq = mgL \sin q \tag{4.2}$$

In such cases, this approach is useful and easy. Let us assume that the stiffness coefficient k is variable. For

$$k > \frac{1}{2}mgL$$

there is only one equilibrium position $q = 0$, and it is stable. For

$$k < \frac{1}{2}mgL$$

the system has three equilibrium positions: $q = 0$ that is unstable, and two equilibrium positions that are stable and determined by the relation

$$2kq = mgL \sin q$$

Now, let us introduce a perturbation of the generalised co-ordinate

$$q = q + \varphi$$

into the potential energy equation, Eq. (4.1), where φ is a perturbation of the position of the system under consideration in the equilibrium point.

Then, the potential energy of the system is as follows

$$E_p = \frac{1}{2}k(q+\varphi)^2 - \frac{1}{2}mgL[1 - \cos(q+\varphi)] \tag{4.3}$$

Expanding the trigonemetric function in Eq. (4.3) into a Taylor series and rejecting the terms of higher orders, we obtain

$$E_p = E_p^n + E_p^p$$

where E_p^n is the nominal potential energy of the system

$$E_p^n = \frac{1}{2}[kq^2 - mgL(1 - \cos q)]$$

whereas E_p^p is the potential energy of the perturbation

$$E_p^p = k\varphi\Big(q + \frac{\varphi}{2}) - \frac{1}{2}mgL\Big[\cos q\Big(\frac{\varphi^2}{2} - \frac{\varphi^4}{24}\Big) + \sin q\Big(\varphi - \frac{\varphi^3}{6} + \frac{\varphi^5}{120}\Big)\Big] \tag{4.4}$$

In the case of the equilibrium point $q = 0$, the perturbation potential energy, Eq. (4.4), has the form

$$E_p^p = \frac{1}{2}\Big[k\varphi^2 - \frac{1}{2}mgL\Big(\varphi^2 - \frac{\varphi^4}{12}\Big)\Big] \tag{4.5}$$

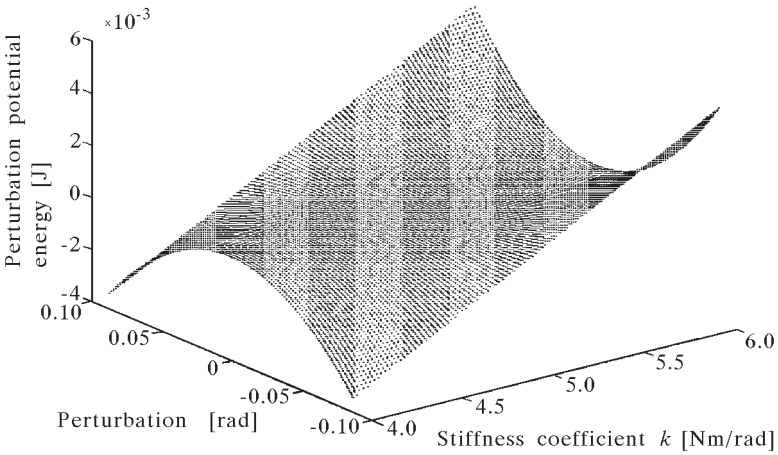

Fig. 7. Transition from the unstable system to the stable one

For given m, L of the system under consideration, the stiffness coefficient k decides whether the critical point is stable or not. In Fig. 7, the distribution of the perturbation potential energy as a function of the stiffness coefficient k and the perturbation of the system generalised co-ordinate, based on Eq. (4.5), is shown.

The following data of the system have been assumed: $m = 1\,\mathrm{kg}$, $L = 1\,\mathrm{m}$.

A transition from the unstable system, $E_p^p < 0$, to the stable one, $E_p^p > 0$, can be seen with an increase in the stiffness k. The boundary stability occurs for the stiffness coefficient $k = 4.905\,\mathrm{Nm/rad}$.

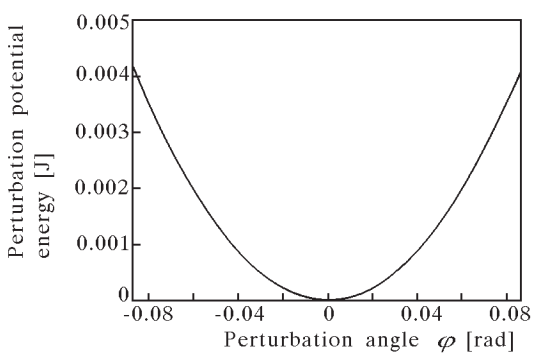

Fig. 8. Stability of the equilibrium $q = 0$

In Fig. 8, changes in the perturbation potential energy in the vicinity of the critical point $q = 0$ for the stiffness $k = 6\,\mathrm{Nm/rad}$ are presented. The perturbation potential energy is positive, so the system is stable in the vicinity of the equilibrium point. In turn, Fig. 9 shows the perturbation potential energy in the vicinity of the equilibrium point $q = 0$ for the stiffness coefficient $k = 4\,\mathrm{Nm/rad}$.

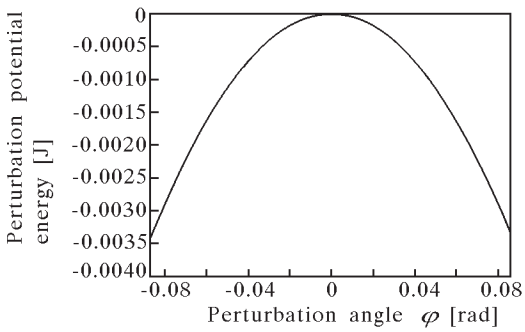

Fig. 9. Unstability of the equilibrium $q = 0$

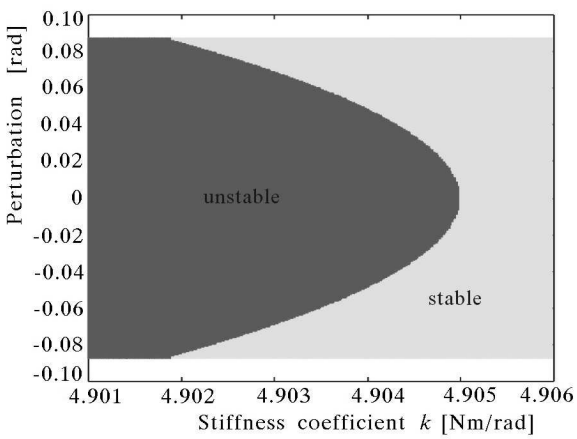

Fig. 10. Identification of the stability boundary for the equilibrium $q = 0$

The perturbation potential energy is negative, so the system is unstable in the critical point. Identification of the stability boundary for the equilibrium point $q = 0$ is performed through an analysis of the change in the perturbation potential energy of the system. In Fig. 10 an energy map of the vicinity of the equilibrium point is shown. It allows for the identification of the stability boundary of the system, which can be stable or unstable, through finding a

value of the parameter at which a bifurcation occurs – a transcritical fork-type bifurcation in the case under consideration. In the figure, the system stability boundary as a function of the position perturbation is shown as well. In the case of dynamical systems with a higher number of degrees of freedom than one, the energy map refers to the energy criterion of stability. In other words, it is the balance of energy needed to bring the dynamical system to the critical point.

For the value of the stiffness coefficient k corresponding to the system stability boundary (boundary value of the stiffness coefficient k between stability and instability of the system), the perturbation potential energy is positive, and thus the system remains stable – Fig. 11.

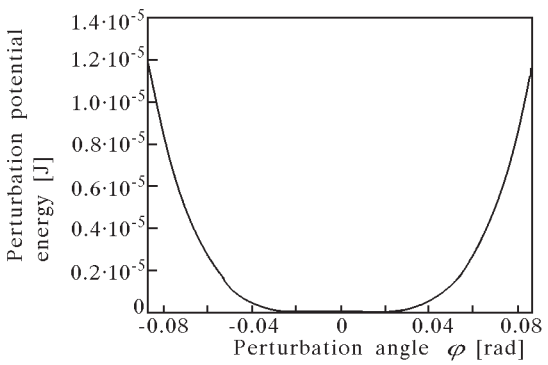

Fig. 11. Perturbation of the system for the coefficient k corresponding to the stability boundary

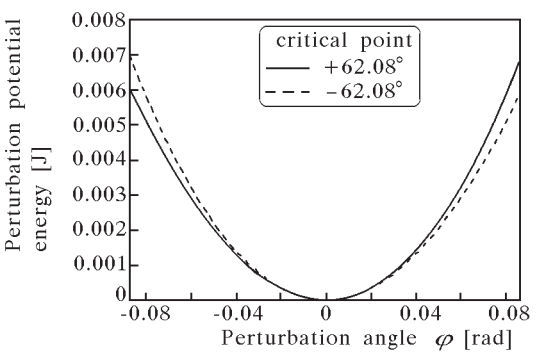

Fig. 12. Stability of the system for the equilibrium $2kq = mgL\sin q$

In turn, changes in the perturbation potential energy as a function of the perturbation in the vicinity of the equilibrium point described by the solution (4.2), for the stiffness coefficient $k = 4\,\mathrm{Nm/rad}$, are presented in Fig. 12.

As the perturbation potential energy is positive, the system remains stable. While drawing a plot of the changes in the perturbation potential energy as a function of the stiffness coefficient k, one can determine the system stability boundary – Fig. 13.

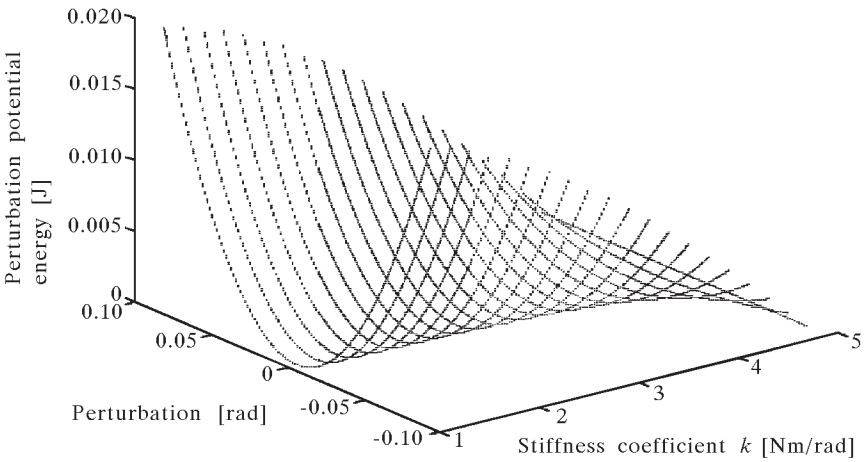

Fig. 13. Finding the stability boundary of the system

It can be seen that the equilibrium point described by the solution (4.2) vanishes for the stiffness coefficient $k = 4.905\,\mathrm{Nm/rad}$.

5. Analysis of characteristic points of an exemplary robot manipulator

Let us consider a robot manipulator with three degrees of freedom as an example. A scheme of such a manipulator is shown in Fig. 14. It is a four-link manipulator with the following kinematic chain structure: R-R-P (R – rotational kinematic pair, P – prismatic kinematic pair). In order to simplify the mathematical model, ideal kinematic pairs of the manipulator have been assumed. In the assumed manipulator model, conservative forces in the form of gravity forces occur.

Now let us consider a relation describing the potential energy of the manipulator in the second stage of motion (motion of the second and third generalised co-ordinate, whereas the first co-ordinate remains stationary).

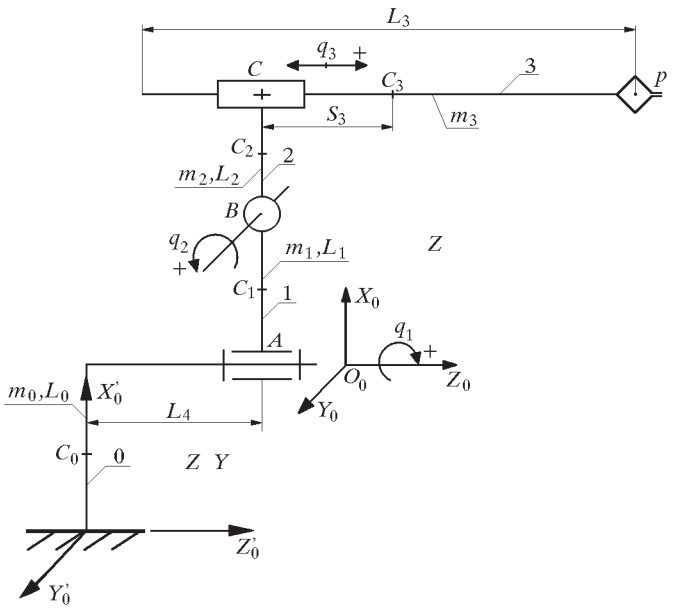

Fig. 14. Scheme of the robot manipulator MAR

Let us introduce perturbations of the co-ordinates. We obtain then a relation describing the manipulator potential energy, in the second stage of motion, in the form

$$E_{p2} = A + Z(\varphi_2, \varphi_3) \tag{5.1}$$

where

A – terms of the potential energy equation describing basic motion; factors depending on the robot configuration and on the mass and geometrical data of its links; in the case under consideration

$$A = \Big(-\frac{1}{2}m_0 L_0 + \frac{1}{2}m_1 L_1 \cos q_1 + m_2 L_1 \cos q_1 + m_3 L_1 \cos q_1)g$$

L_i – length of the ith manipulator link

m_i – mass of the ith manipulator link

Z – terms of the potential energy equation describing the manipulator perturbation

or transforming Eq. (5.1)

$$E_{p2} = A + B\cos(q_2 + \varphi_2) + C\sin(q_2 + \varphi_2) + D\varphi_3 \sin(q_2 + \varphi_2) \tag{5.2}$$

where

$$\begin{aligned} B &= \Big(\frac{1}{2}m_2 + m_3\Big)L_2 g\cos q_1 \\ C &= m_3(s_3 + q_3)g\cos q_1 \\ D &= m_3 g\cos q_1 \end{aligned} \tag{5.3}$$

and s_3 is the initial position of the centre of gravity for link 3, with regard to the position of point C – Fig. 14.

Thus, when the perturbation occurs, we can express the manipulator potential energy as a sum of the manipulator nominal potential energy and of the potential energy, called the potential energy of the manipulator perturbation, i.e., generally in the form

$$E_{p2} = E_{p2}^n + E_{p2}^p \tag{5.4}$$

After some mathematical transformations and taking into account the close vicinity of manipulator equilibrium points, the trigonometric functions are replaced by a Taylor series, and the series terms higher than four are rejected. Then, on the basis of Eq. (5.1), the potential energy of the system perturbations is

$$\begin{aligned} E_{p2}^p &= -(B\cos q_2 + C\sin q_2)\Big(\frac{\varphi_2^2}{2} - \frac{\varphi_2^4}{24}\Big) + \\ &- (B\sin q_2 - C\cos q_2 - D\varphi_3\cos q_2)\Big(\varphi_2 - \frac{\varphi_2^3}{6} + \frac{\varphi_2^5}{120}\Big) + \\ &+ D\varphi_3\sin q_2\Big(1 - \frac{\varphi_2^2}{2} + \frac{\varphi_2^4}{24}\Big) \end{aligned} \tag{5.5}$$

If we assume the values of the parameters of individual equilibrium points, then on the basis of Eq. (5.3), we obtain the values of the coefficients B, C, D of Eq. (5.5) in the neighbourhood of manipulator individual equilibrium points. A collection of the obtained results is presented in Table 2.

Employing the data included in Table 2, Eq. (5.5) assumes the form

$$\begin{aligned} E_{p2}^p = &-\frac{B\sin q_2}{120}\varphi_2^5 + \frac{B\cos q_2}{24}\varphi_2^4 + \frac{B\sin q_2}{6}\varphi_2^3 - \frac{B\cos q_2}{2}\varphi_2^2 + \\ &-B\sin q_2\varphi_2 + D\sin q_2\varphi_3 + \frac{D\cos q_2}{120}\varphi_2^5\varphi_3 + \frac{D\sin q_2}{24}\varphi_2^4\varphi_3 + \\ &-\frac{D\cos q_2}{6}\varphi_2^3\varphi_3 - \frac{D\sin q_2}{2}\varphi_2^2\varphi_3 + D\cos q_2\varphi_2\varphi_3 \end{aligned} \tag{5.6}$$

Table 2

Maniulator configuration	Parameters of the critical point in kinematic pairs	Coefficient		
		B	C	D
High	$q_1 = 0$ $q_2 = 0$ $q_3 = -s_3$	$(m_2/2 + m_3)L_2 g$	0	$m_3 g$
High	$q_1 = 0$ $q_2 = \pi$ $q_3 = -s_3$	$(m_2/2 + m_3)L_2 g$	0	$m_3 g$
Low	$q_1 = \pi$ $q_2 = 0$ $q_3 = -s_3$	$-(m_2/2 + m_3)L_2 g$	0	$-m_3 g$
Low	$q_1 = \pi$ $q_2 = \pi$ $q_3 = -s_3$	$-(m_2/2 + m_3)L_2 g$	0	$-m_3 g$

Equation (5.6) allows for an analysis of changes of the manipulator perturbation potential energy as a function of the perturbation magnitude of individual degrees of freedom. We can distinguish here the cases mentioned in the theoretical introduction.

In the case of a high configuration of the manipulator, and at the generalised co-ordinate $q_2 = 0$ as well as in the case of a low configuration of the manipulator, and at the generalised co-ordinate $q_2 = \pi$, Eq. (5.6) assumes the form

$$E_{p2}^{p} = \frac{1}{2} B \varphi_2^2 \Big(\frac{1}{12} \varphi_2^2 - 1 \Big) + D \varphi_2 \varphi_3 \Big(\frac{1}{120} \varphi_2^4 \varphi_3 - \frac{1}{6} \varphi_2^2 + 1 \Big) \tag{5.7}$$

In the case of a high configuration of the manipulator, and at the generalised co-ordinate $q_2 = \pi$ as well as in the case of a low configuration of the manipulator, and at the generalised co-ordinate $q_2 = 0$, Eq. (5.6) assumes, in turn, the form

$$E_{p2}^{p} = \frac{1}{2} B \varphi_2^2 \Big(1 - \frac{1}{12} \varphi_2^2 \Big) - D \varphi_2 \varphi_3 \Big(\frac{1}{120} \varphi_2^4 \varphi_3 - \frac{1}{6} \varphi_2^2 + 1 \Big) \tag{5.8}$$

6. Examples of numerical analysis results

Analysing Fig. 15-Fig. 18, one can observe changes in the potential energy of the manipulator location perturbation as a function of the perturbation magnitude, and the case when:

- a perturbation of one of the generalised co-ordinates of the manipulator occurs – in this case it is the generalised co-ordinate q_2, see Fig. 15 and Fig. 16
- perturbations of a higher number of the generalised co-ordinates occur – in the considered case these are q_2 and q_3, see Fig. 17 and Fig. 18.

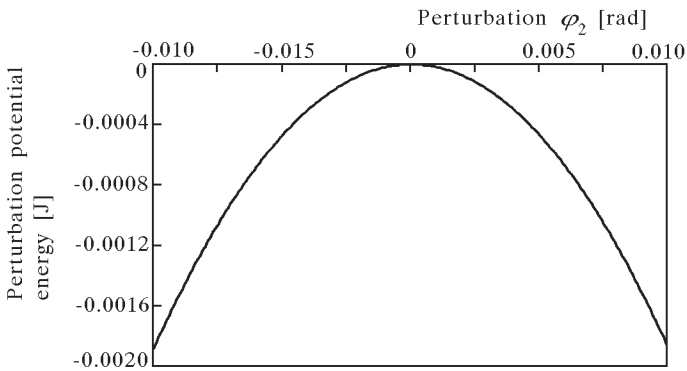

Fig. 15. Perturbation potential energy distribution in the case of an unstable point

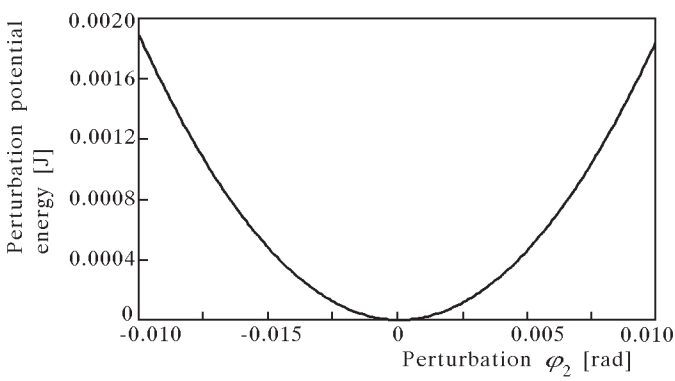

Fig. 16. Perturbation potential energy distribution in the case of a stable point

In Fig. 17 and Fig. 18 a distribution of the perturbation potential energy as a function of the two-dimensional perturbation φ_2 and φ_3 for selected cases of equilibrium points is presented.

The perturbations of the location of the manipulator links in the following ranges

$$\varphi_2 = \pm 0.01 \text{ rad} \qquad \varphi_3 = \pm 0.0005 \text{ rad}$$

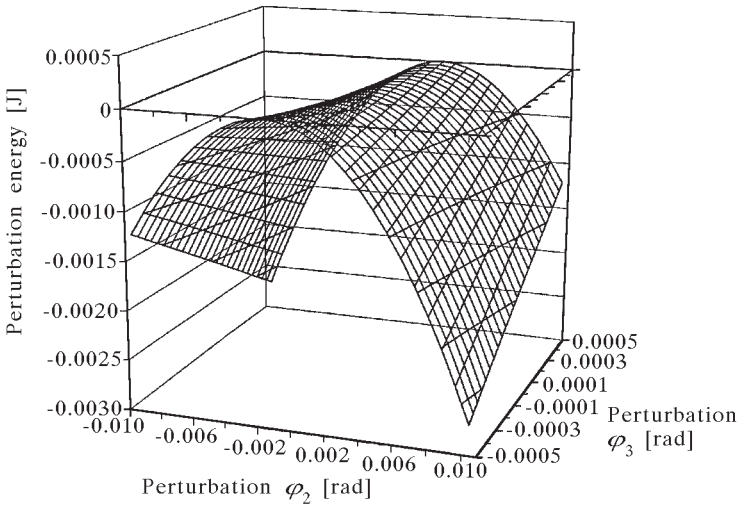

Fig. 17. Perturbation potential energy space distribution in the case of an unstable point

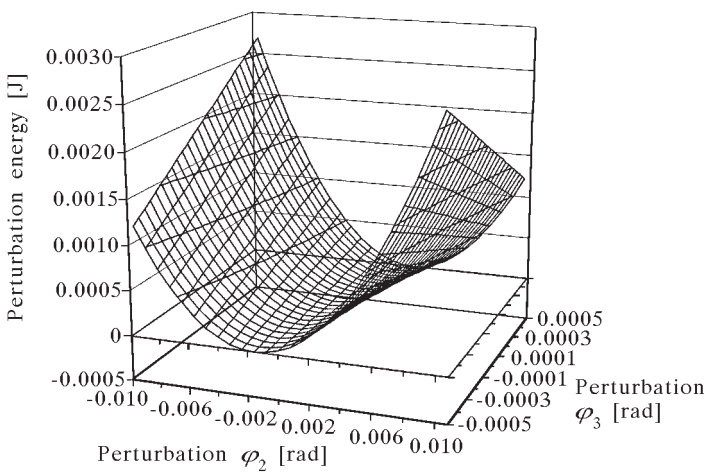

Fig. 18. Perturbation potential energy space distribution in the case of a stable point

have been introduced into the system under analysis, [Asada 1986].

Taking into account the stability criteria mentioned in theoretical introduction, the types of individual critical points of the considered manipulator are presented in Table 3.

Table 3

Distribution of perturbation potential energy	Kind of critical point
Figure 15	Unstable point (anti-attractor)
Figure 16	Stable point (attractor)
Figure 17	Energy and position unstable point
Figure 18	Energy and position stable point

The notion of an attractor is understood as the effect of "attraction" of a trajectory of changes of the perturbation potential energy by the critical point, whereas the notion of an anti-attractor refers to the property of the critical point consisting in "repulsing" the trajectory of the perturbation potential energy changes.

The analysis of Eqs (5.7) and (5.8) and of Fig. 17 and Fig. 18 allows one to observe that in the case of lack of the perturbation of the critical point of the second generalised co-ordinate a perturbation of the critical point related to the third co-ordinate does not affect the magnitude of the perturbation potential energy of the manipulator. Thus, one can state that the behaviour of the system in the neighbourhood of the critical point is influenced by the perturbation of the second generalised co-ordinate of the manipulator.

In Fig. 15 and Fig. 16, the character of the potential energy distribution of the manipulator perturbation for the case of a zero perturbation of the manipulator third generalised co-ordinate in vicinities of various types of critical points, in the second stage of robot motion, is presented. The figures confirm the obtained results referring to the types of individual critical points determined by the method of eigenvalues and by the motion perturbation method (Kapitaniak and Wojewoda, 2001; Szumiński, 1995, 1997).

Table 4

		Manipulator link number i		
		1	2	3
Kind of kinematic pair		R-rotational	R-rotational	P-prismatic
Link mass m_i [kg]		12.7	12.7	15
Link lenght L_i [m]		0.17	0.18	0.42
Position of	x^r_{si} [m]	$-L_1/2$	$-L_2/2$	0
the centre	y^r_{si} [m]	0	0	0
of gravity	z^r_{si} [m]	0	0	$-L_3/2$
$S_3 = 0.18$ m				

Stability simulations for the cases of the determined critical points of the low configuration of the manipulator $(q_1 = \pi)$ yield analogous results to the presented simulation for the high configuration $(q_1 = 0)$.

Physical and geometrical data of the analysed manipulator are included in Table 4.

7. Conclusions

The presented method of the potential energy perturbation of a dynamical system allows for the analysis of position and energy stability in the critical point vicinity as a function of the perturbation of an arbitrary number of the system generalised co-ordinates. It enables simulation of the dynamical behaviour for various cases of generalised co-ordinate perturbations, including the effect of perturbations of individual generalised co-ordinates on the dynamical system stability in the critical point. Formulation of equations of motion of the system is not required. The method is based on the analysis of changes in the potential energy of the system upset from the equilibrium position.

The forms and kinds of critical points are closely related to the manipulator model, which has been assumed in the investigations.

This method can be applied for the determination of control algorithms of motion in terms of subsequent generalised co-ordinates of the dynamical system in order to maintain the conditions of its position and energy stability in the critical point.

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Metoda energetyczna analizy stateczności punktów krytycznych manipulatorów

Streszczenie

W artykule przedstawiono metodę energetyczną analizy stateczności punktów krytycznych (położeń równowagi) układów dynamicznych o dowolnej skończonej liczbie stopni swobody. Proponowana metoda polega na analizie bilansu energii potencjalnej zaburzenia dowolnej liczby współrzędnych uogólnionych układu dynamicznego w celu okreslenia stateczności energetycznej poszczególnych stopni swobody oraz całego

układu. W przypadku układów mechanicznych przedstawiono kryterium stateczności w sensie minimalizacji obciążeń układów napędowych siłami zachowawczymi. Przedstawiono przykłady zastosowania metody dla celów analizy stateczności punktów krytycznych układów o jednym i dwóch stoopniach swobody.

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