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# Chaos synchronization and hyperchaos

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**Abstract.** We discuss the relation between phenomena of chaos synchronization in coupled systems and creation of hyperchaotic attractors (attractors with at least two positive Lyapunov exponents. Such attractors are common in higher-dimensional dynamical systems (at least two-dimensional maps or four-dimensional flows). Riddling bifurcation i.e., the bifurcation in which one of the unstable periodic orbits embedded in a chaotic attractor located on the invariant manifold becomes unstable transversely to the attractor leads to the loss of chaos synchronization in coupled identical systems.

We show that generalized riddling bifurcation defined as the bifurcation in which one of the unstable periodic orbits embedded in a higher-dimensional chaotic attractor (not necessarily located on the invariant manifold) becomes unstable transversely to the attractor explains mechanism of the creation of hyperchaotic attractors. Additionally we show that the generalized riddling bifurcation can give physical mechanism explaining interstellar journeys described in science-fiction literature.

## 1. Introduction

Nowadays we can observe a growing interest in the higher-dimensional dynamical systems as the result of extensive studies of the techniques to control [1] and synchronize [2] chaotic systems. If the map is at least two-dimensional or a flow is at least four-dimensional its evolution can take place on the hyperchaotic attractor. Such attractors are characterized by at least two positive Lyapunov exponents for typical trajectories on them. The first example of such a system was presented by Rossler [3] for a model of particular chemical reaction. Later, the hyperchaotic attractors have been found in electronic circuits and other chemical reactions [4]. In the works [5] it was shown that by weakly coupling  $n$  chaotic system it is possible to obtain hyperchaotic attractor with  $n$  positive Lyapunov exponents. The transition from chaos to hyperchaos has been studied in [6]. It was shown that at this transition the attractor's dimension and the second Lyapunov exponent grow continuously.

Unstable periodic orbits (UPOs) constitute one of the most basic invariants of a dynamical system [7]. The infinite number of UPOs embedded in a chaotic set provides the skeleton of the attractor and it allows for the characterization and the estimation in a fundamental way of many dynamical invariants such as the natural measure, the spectra of Lyapunov exponents and the fractal dimension [8]. UPOs play a fundamental role in the mechanism of destabilization of the chaotic attractor localized in some symmetric invariant manifold and it is responsible for the dynamics of phenomena such as riddling of the basin of attraction and bubbling of the chaotic attractor [9]. Recently, UPOs have also been used in the description of higher-dimensional dynamical phenomena of chaos-hyperchaos transition [10]. It has been shown that the chaos-hyperchaos transition, as well as the blowout bifurcation, is mediated by an infinite number of

UPOs which become unstable in at least two directions in the neighborhood of the transition point. The simultaneous existence of UPOs with different number of unstable direction gives rise to a new kind of nonhyperbolicity known as unstable dimension variability [11] and it may give a possible dynamic mechanism for the smooth transition through zero of the second Lyapunov exponent.

In this paper, we argue that the phenomena characteristic of one-dimensional attractors located on the invariant manifold, like riddling and bubbling, can be generalized to higher-dimensional attractors. We show that the riddling, which occurs after the appearance of the first UPO with more than one unstable direction in the chaotic attractor on the invariant manifold allows for the growth of the attractor by the bursting along the new unstable direction. We point out that this is a typical way by which higher-dimensional attractors grow.

## 2. Riddling of one-dimensional attractors

Two identical chaotic systems  $x_{n+1} = f(x_n)$  and  $y_{n+1} = f(y_n)$ ,  $x, y \in \mathcal{R}$  evolving on an asymptotically stable chaotic attractor  $A$ , when coupled as

$$\begin{aligned}x_{n+1} &= f(x_n) + d_1(y_n - x_n), \\y_{n+1} &= f(y_n) + d_2(x_n - y_n),\end{aligned}\tag{1}$$

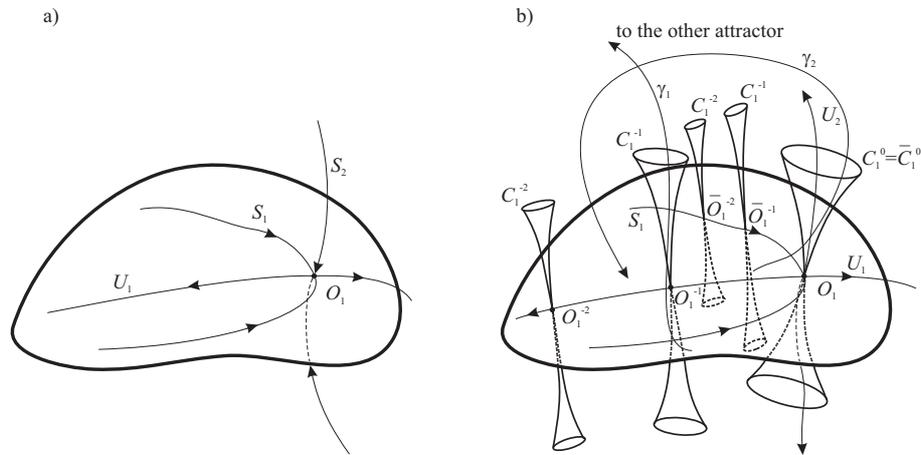
can be synchronized for some ranges of  $d_{1,2} \in \mathcal{R}$ , i.e.,  $|x_n - y_n| \rightarrow 0$  as  $n \rightarrow \infty$  [2]. In the complete synchronized regime, the dynamics of the coupled system (1) is restricted to one-dimensional invariant subspace  $x_n = y_n$ , so the problem of synchronization of chaotic systems can be understood as a problem of stability of one-dimensional chaotic attractor  $A$  in two-dimensional phase space.

The basin of attraction  $\beta(A)$  is the set of points whose  $\omega$ -limit set is contained in  $A$ . In Milnor's definition [12] of an attractor, the basin of attraction needs not to include the whole neighborhood of the attractor, i.e., we say that  $A$  is a weak Milnor attractor if  $\beta(A)$  has positive Lebesgue measure. For example, a riddled basin [13], which has recently been found to be typical for a certain class of dynamical systems with one-dimensional invariant subspace [like  $x_n = y_n$  in the example (1)], has positive Lebesgue measure but does not contain any open neighborhood. In this case, the basin of attraction  $\beta(A)$  may be a fat fractal so that any neighborhood of the attractor intersects its own basin with positive measure, but it intersects the basin of another attractor also with positive measure.

Dynamics of the system (1) is described by two Lyapunov exponents. One of them describes the evolution on the invariant manifold  $x = y$  and is always positive. The second exponent characterizes evolution transverse to this manifold and it is called the transversal exponent. If the transversal Lyapunov exponent is negative, the set  $A$  is an attractor, at least in the weak Milnor sense.

It may happen that, though the transversal Lyapunov exponent is negative, there exist trajectories in the attractor  $A$  which are transversally repelling. In this case  $A$  is a Milnor attractor with *locally riddled* basin, i.e., there is an open neighborhood  $U$  of  $A$  such that in any neighborhood  $V$  of any point in  $A$ , there is a set of points in  $V \cap U$  of positive measure which leaves  $U$  in a finite time. The trajectories which leave the neighborhood  $U$  can either go to the other attractor (attractors) or after a finite number of iterations be diverted back to  $A$ . The latter case is also known as *bubbling* of attractor  $A$ .

Transition from asymptotically stable attractor to the Milnor attractor with riddled basin occurs via riddling bifurcation [14] in which one of the UPO's (say  $O_1$ ) embedded in the attractor  $A$  becomes transversally unstable. (It becomes an unstable repelling node.) This transverse instability allows the trajectories near the attractor  $A$  to escape. In the neighborhood of  $O_1$ , two tongues of points that do not belong to the basin of attraction of the attractor in the



**Figure 1.** The mechanism which allows bubbling of higher-dimensional attractors; (a) before the riddling bifurcation, (b) after the riddling bifurcation

invariant manifold are developed. Moreover, each preimage of  $O_1$  also develops such tongues. Since preimages of  $O_1$  are dense in the invariant manifold, an infinite number of tongues is created simultaneously.

### 3. Riddling bifurcation of higher-dimensional attractors

In the previous paper [10], we studied the dynamical system given by a dissipative map  $u_{n+1} = f(a, u_n)$ , where  $u \in \mathcal{R}^2$  and  $a \in \mathcal{R}$ . In such system, due to the stretching and folding mechanism, one can observe attractors with one or two positive Lyapunov exponents. Generally, if such a map is  $N$ -dimensional ( $u \in \mathcal{R}^N$ ) one can observe attractors with  $N$  positive Lyapunov exponents. We assumed that the system evolved on the chaotic attractor  $A$  (i.e., with one positive Lyapunov exponent) and allowed the control parameter to vary slowly in such a way that the second Lyapunov exponent became positive and thus the attractor  $A$  became hyperchaotic. We gave evidence [10] that the bifurcations of UPO which are characteristic for the chaos-hyperchaos transition are typically stretching (spreading) in a given control parameter interval, and that the transition mechanism has the same characteristic features as the blowout bifurcation of the attractors located in an invariant manifold in systems with symmetry [9]. It was shown that the transition to hyperchaos is initiated when a repeller arises in the attractor through one of the following situations:

- (i) some usually low-period periodic orbit  $\gamma$ , embedded in the chaotic attractor  $A$  undergoes a saddle-repeller bifurcation,
- (ii) a repelling node in the attractor appears in a saddle-node bifurcation,
- (iii) the repeller (unstable node or focus) originally located off the attractor is absorbed by the expanding attractor.

Consider the chaotic attractor  $A$  located in a 3-dimensional phase space, as shown in Fig. 1, and we denote one of the UPOs embedded in it by  $O_1$ . Before the bifurcation,  $O_1$  has stable  $S_1$  and unstable  $U_1$  manifolds located on the attractor  $A$  and stable manifold  $S_2$  transverse to  $A$  [Fig.1(a)]. (Most of the known chaotic attractors (for example Lorenz and Rossler), in the macroscopic approximation, has "an attractor manifold" where the limiting dynamics is governed.) After the bifurcation the manifold  $S_2$  transverse to  $A$  becomes unstable. In Fig.1(b) it is denoted by  $U_2$ .

If the considered map is non-invertible appearance of the first UPO with more than one unstable direction on the attractor  $A$  creates the tongues  $C_1, C_1^{-1}, C_1^{-2}, \dots$  anchored respectively at  $O_1$  and at all preimages of  $O_1$  (denoted by  $O_1^{-1}, O_1^{-2}, \dots$ ) on the attractor  $A$  with such a property that all points in these sets leave the neighborhood of  $A$ . (In any open neighborhood  $U$  of  $A$  there is a positive measure set of points which leave this neighborhood.) The system trajectories entering the neighborhood of  $O_1$  or neighborhoods of all preimages of  $O_1$  located on the closure of the unstable manifold  $U_1$  on the attractor  $A$  leave the attractor along the unstable manifold ( $U_2$ ) which is transverse to  $A$ . The trajectory which leaves the attractor  $A$ , could be asymptotic to the other attractor  $B$  [trajectory  $\gamma_1$  in Fig.1(b)] and in this case the basin of attractor  $A$  is riddled.

If the considered system is invertible there exists orbit  $\{\bar{O}_1^n\}_{n=-\infty}^{n=\infty} \in S_1 \in A$  approaching  $O_1$  as  $t \rightarrow \infty$ . At the points  $\bar{O}_1^n$  the second sequence of tongues  $\bar{C}_1, \bar{C}_1^{-1}, \bar{C}_1^{-2}, \dots$  is anchored as can be seen in Fig.1(b). The fate of any trajectory entering these tongues is the same as described for tongues  $C_1, C_1^{-1}, C_1^{-2}, \dots$  in the case of non-invertible system.

Riddling of  $m$ -dimensional attractor  $A$  ( $m > 1$ ) can be defined as follows: *basin of attraction  $\beta(A)$  of attractor  $A$  is called riddled if there exists an infinite set  $R \subset A$  with such a property that in any open neighborhood of  $R$  there exist points which do not belong to  $\beta(A)$ .*

In the example shown in Fig.1, the set  $R$  is given by  $O_1 \cup O_1^{-1} \cup O_1^{-2} \cup \dots$  or  $\bar{O}_1 \cup \bar{O}_1^{-1} \cup \bar{O}_1^{-2} \cup \dots$ . As the basin of attraction  $\beta(A)$  becomes riddled, after the appearance of the first transversely UPO in the attractor  $A$ , we propose to call the bifurcation in which such UPO is created as the *generalized riddling bifurcation*.

If the attractor  $B$  does not exist, such a trajectory has to come back to the attractor  $A$  [trajectory  $\gamma_2$  in Fig.1(b)]. In this case, the riddling is only local but the bursts off the attractor  $A$  allows it to grow in the direction which was not allowed before the appearance of  $O_1$ . We propose to call this phenomenon the *higher-dimensional bubbling*.

In the case of the riddling of higher-dimensional attractors the set  $R = \{O_1, O_1^{-1}, O_1^{-2}, \dots\}$  or  $R = \{\bar{O}_1, \bar{O}_1^{-1}, \bar{O}_1^{-2}, \dots\}$  is countably infinite. Due to the ergodicity, any trajectory  $\gamma$  on the attractor  $A$  has to visit the neighborhood of  $O_1$  (or one of  $O_1^{-1}, O_1^{-2}, \dots, \bar{O}_1^{-1}, \bar{O}_1^{-2}, \dots$ ) and, if perturbed off the attractor  $A$ , it leave the attractor in a finite time.

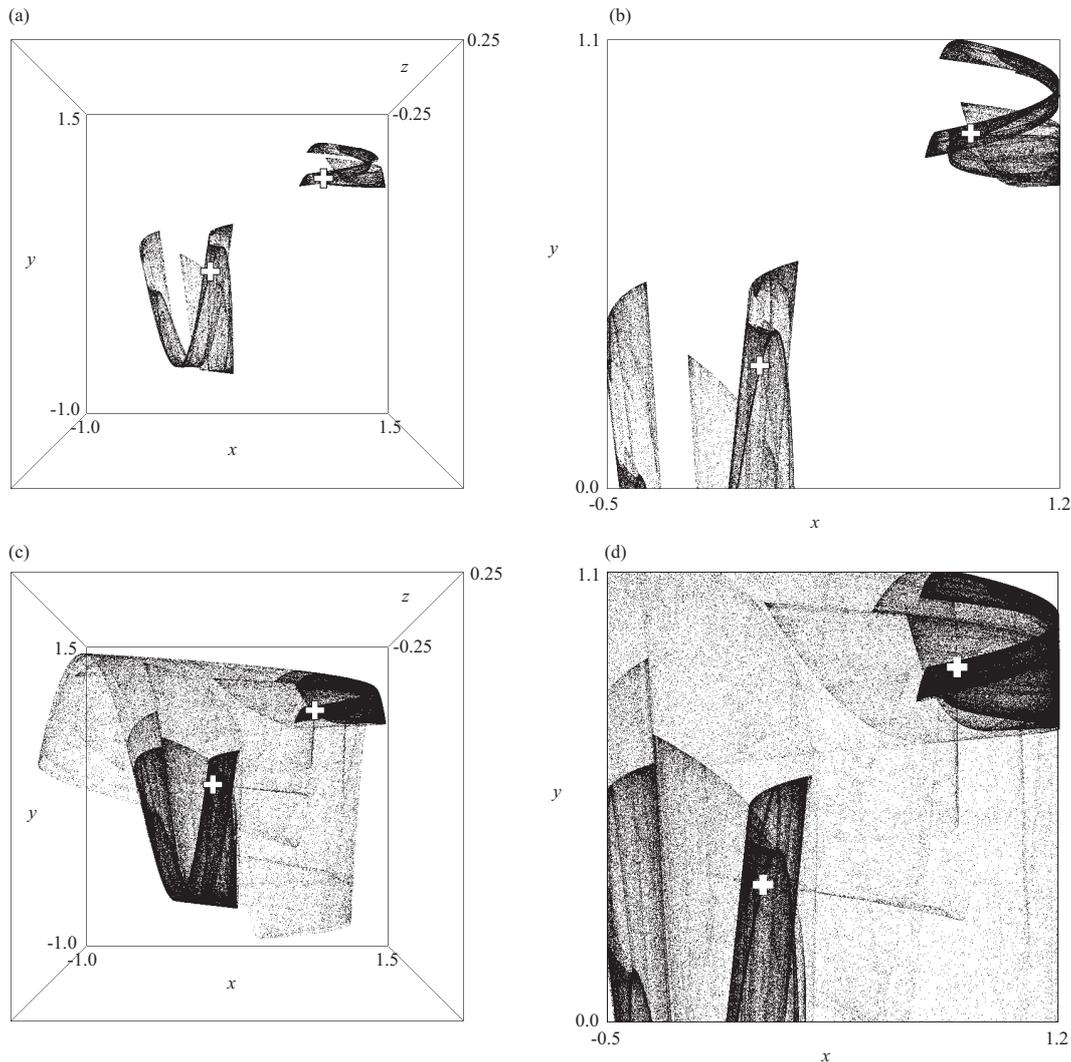
### 3.1. Example

As an example, consider a three-dimensional map  $F$  in the form:

$$\begin{aligned} x_{n+1} &= 1 + z_n - ay_n^2, \\ y_{n+1} &= 1 + by_n - ax_n^2, \\ z_{n+1} &= bx_n \end{aligned}$$

where  $x_n, y_n, z_n \in \mathcal{R}$  are dynamical variables,  $a$  and  $b \neq 0$  are the system parameters. This map was introduced as an example of a simple system which shows both chaotic and hyperchaotic behaviour. This map is invertible as its Jacobian is equal  $-b^2$ .

In our numerical simulations, we consider  $b = 0.2$  and take  $a$  as a control parameter. For  $a < 1.267$ , map (2) has a chaotic attractor  $A$  with such a property that all UPO embedded in it have stable transverse directions. At  $a = 1.2678$ , the first UPO with unstable transverse direction appears initiating the sequence of bifurcations of UPOs which lead to the chaos-hyperchaos transition at  $a = 1.297$ . The example of the chaotic attractor  $A$  ( $a = 1.26$ ) is shown in Fig. 2(a,b). The crosses in Fig. 2(a,b) indicate a period-2 UPO with one stable  $S_1$  and one unstable  $U_1$  directions along the attractor  $A$  and the stable direction  $S_2$  transverse to  $A$ . By increasing the parameter  $a$  this period-2 UPO undergoes the bifurcation in which its transverse direction becomes unstable ( $S_2$  becomes  $U_2$ ). After the bifurcation one can observe tongues of



**Figure 2.** Attractor of the map (2) for  $b = 0.2$  (a) before the riddling bifurcation  $a = 1.26$ , (b) enlargement of (a), (c) after the riddling bifurcation  $a = 1.27$ , (d) enlargement of (c)

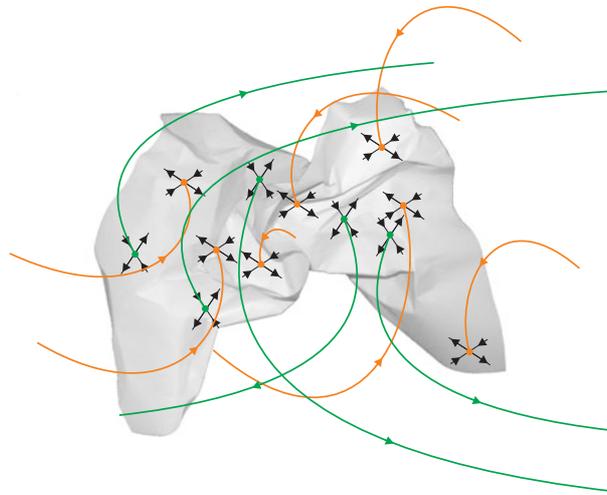
points leaving the neighborhood of the attractor  $A$  in the transverse direction  $U_2$ , as shown in Fig. 2(c,d). These tongues anchor at  $O_1$  and at the points  $\bar{O}_1^{-1}, \bar{O}_1^{-2}, \dots$  on the stable manifold  $S_1$ .

The basin of attractor  $A$  is riddled in the sense of definition introduced in the Sec. III. As  $A$  is the only attractor of the map (2) in the considered range of parameters, the trajectories leaving its neighborhood have to come back to it. We observe the phenomenon of the higher-dimensional bubbling.

#### 4. Riddling bifurcation and ... interstellar journeys

In this section we argue that the transition from lower-dimensional attractor to the higher-dimensional one can explain the way in which interstellar journeys are described in science-fiction literature.

Consider the  $n$ -dimensional chaotic attractor  $A$  located in a  $m$ -dimensional phase space ( $n < m$ ), as shown in Fig. 3. Consider that attractor  $A$  is a simple model of the attractors

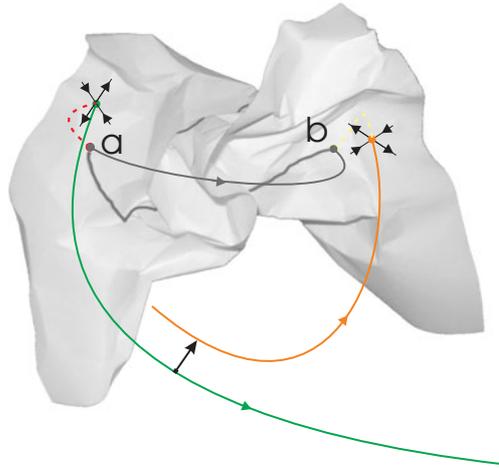


**Figure 3.** Chaotic attractor  $A$  with UPO with stable (red) and unstable (green) transverse manifolds

located in the El Naschie's Cantorian spacetime [16]. Assume that UPOs embedded in the attractor have already undergone riddling bifurcation [14] so in the attractor two types of UPOs are embedded. The first type are the orbits with stable transverse manifold (indicated in red colour in Fig. 3) and the second type these with unstable transverse manifold (indicated in green). Suppose that the phase space trajectory on the attractor  $A$  is close to the point  $a$  and that we have to implement the control procedure which allows us to go from the point  $a$  to the point  $b \in A$  (Fig. 4). The straightforward way is to restrict the path from  $a$  to  $b$  to the attractor  $A$  (navy blue line in Fig. 4). Due to the ergodicity point  $b$  will be reached in finite time but this could be too long for practical acceptance. The alternative approach is the path in the neighborhood of the attractor  $A$  like the black line in Fig. 4. In this case the phase space trajectory has to leave the attractor  $A$ , stay in its neighborhood and return to the attractor in the appropriate point. Dynamically such a situation is possible when in the neighborhood of the point  $a$  there exists UPO with unstable transverse manifold then by applying control one can move the trajectory to the neighborhood of this UPO (along the broken red line in Fig. 4) and allow it to leave the attractor (along the green line in Fig. 4). After leaving the attractor the trajectory has to come close to the stable manifold of the other UPO with stable transverse manifold (red line in Fig. 4). At this point the control moving the trajectory to the stable manifold has to be applied. Along the green manifold the trajectory returns to the attractor in the neighborhood of the point  $b$ . After another application of the control the trajectory reaches the target point  $b$  (yellow line in Fig. 4).

Now recall one of the fundamental problems in the discussion of the possible interstellar journeys. Due to the huge distance in the universe and speed limitations of the space crafts, the journeys to the other stellar systems seem to be impossible (at least at the current state of science and technology) carried or even imagine. On the other hand, there are plenty of descriptions of these journeys in science-fiction literature (for example [15]. In the great number of novels such journeys are possible due to the existence of super- (or extra-) space which exists besides the universe in which human beings are living. Huge distances are covered by entering this super-space and returning to the home universe in the appropriate point. Usually there are some limitations:

- (i) super-space is not reachable from any point of the universe but only from the neighborhood



**Figure 4.** Controlling procedure

of some special points,

- (ii) the energy is necessary for entering and leaving super-space,
- (iii) the return to the target point of the home universe is not always possible but the return in the neighborhood of the target is possible.

Now assume that

- (i) the points which allow entering super-space are UPOs with transversally unstable manifolds,
- (ii) the return to the universe is possible along the transversally stable manifold of the appropriate UPO,
- (iii) the energy is necessary to reach the neighborhood of appropriate UPO and to change manifolds in the super-space.

At this point, one immediately finds out the analogy between the controlling procedure based on Fig. 4 and interstellar journeys (at least these described in science-fiction literature). The existence of the points in the universe which underwent riddling bifurcation (black holes,....??) is the necessary condition for interstellar journeys.

## 5. Conclusions

In summary, we showed the analogy between the phenomena connected with chaos synchronization and the appearance of higher-dimensional hyperchaotic attractors. The appearance of the first UPO with more than one unstable directions on the chaotic attractor allows the trajectory evolving in a neighborhood of the attractor to leave it along the direction transverse to the attractor. Trajectory which leaves the attractor, can be attracted by another attractor and this phenomena we described as generalized riddling. If the other attractor does not exist, such a trajectory comes back to the attractor and one has bubbling, i.e., the trajectory bursts in the direction transverse to the attractor along the invariant manifold. It seems that the bubbling of the higher-dimensional attractor is a typical way in which attractors grow.

Additionally we proposed the controlling procedure to move the system trajectory from the point on the attractor to the other target attractor point, which explore the higher-dimensionality of the phase space.

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