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Physics Letters A 334 (2005) 169–172
PHYSICS LETTERS A
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Frequency synchronization of clusters in coupled extended systems

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Received 2 April 2004; received in revised form 27 September 2004; accepted 3 November 2004

Available online 13 November 2004

Communicated by A.P. Fordy

Abstract

In this Letter we study an effect of synchronization of clusters with the same average frequency in interacting extended systems. We show that the type of sychronization is a consequence of frequency locking in each point of the system.

Inhomogeneity is a common property of real distributed systems and plays an essential role in their behavior [1,2]. Dependence of a system parameter on spatial coordinates often leads to formation of cluster structures. A cluster is considered to be a group of oscillators (in spatially discrete systems) or a region in continued systems which is characterized by some constant (or almost constant) quantitative characteristic of oscillations. Distributed self-sustained oscillatory systems with natural frequencies varying along a spatial coordinate can serve as models for different phenomena in physics, chemistry and biology [1–4]. Such systems can demonstrate a phenomenon of partial synchronization which manifests itself in occurring of regions with equal average frequencies, which are called “frequency clusters”. This type of partial synchronization can be called frequency synchronization (FS). The frequency clusters have been found both in inhomogeneous chains of self-sustained oscillators [5,7,8] and in self-sustained oscillating media [6,9].

Two distributed systems can demonstrate an effect of mutual synchronization when they interact. Effects of synchronization of spatio-temporal behavior of interacting distributed systems are described in a number of publications [10–13]. However, the problem of mutual synchronization of active media in the regime of frequency clusters has not been considered in the literature up fill now.

In the present work we study an effect of synchronization of frequency clusters in interacting systems and determine FS region on the plane of control parameters. We argue that FS of clusters in spatial structures is a direct consequence of the phenomenon of frequency locking which occurs in each point of the

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2004.11.005

system. As an example of extended system we consider the Ginzburg–Landau equation in the following form:

$$a_t = i\omega(x)a + \frac{1}{2}(1 - |a|^2)a + ga_{xx}, \tag{1}$$

where $i = \sqrt{-1}$; and $a(x,t)$ is the complex amplitude of oscillations that depends on time t and the spatial variable x, $a_t = \partial a/\partial t$, $a_{xx} = \partial^2 a/\partial x^2$. The function $\omega(x)$ assigns inhomogeneity of the distribution of natural frequency along the spatial variable x. We use the linear dependence $\omega(x) = \Delta x/l$, where l is the length of the system, and Δ is the parameter of the frequency mismatch in the boundary points of the system. The diffusion coefficient is supposed to have real values only, i.e., we neglect spatial interactions of the reactive type. The chosen model corresponds to an inhomogeneous chain of quasi-harmonic oscillators [7] under limited transition to a continuous spatial coordinate. We explore the system with finite length l and free boundary conditions: $a_x(x,t)|_{x=0;l} = 0$. Initial conditions are chosen randomly near a homogeneous state.

Two systems (1) coupled in the following way:

$$a_t = i\omega_1(x)a + \frac{1}{2}(1 - |a|^2)a + g_1 a_{xx} + \varepsilon(b - a),$$
$$b_t = i\omega_2(x)b + \frac{1}{2}(1 - |b|^2)b + g_2 b_{xx} + \varepsilon(a - b), \tag{2}$$

where $a(x,t)$ and $b(x,t)$ are complex amplitudes of the first and the second system, and ε is the coupling parameter interact in each point of the space. Eq. (2) were numerically integrated with a implicit scheme of forward and backward iterations [14]. For both systems we calculated the dependence of the average frequency of oscillations in each point of the system

$$\Omega_{1,2} = \left\langle \frac{d\phi_{1,2}}{dt} \right\rangle$$

on the spatial coordinate x. Here ϕ is the instantaneous phase, $\phi \in (-\infty; \infty)$, $\langle\,\rangle$ denotes time averaging.

We start with considering the behavior of the single system (1). When the mismatch is absent, $\Delta = 0$, Eq. (1) has a homogeneous stationary solution which corresponds to a standing wave regime in the homogeneous system. The frequency mismatch induces distortion of the phase of oscillations along the spatial coordinate x. While Δ is small, the stationary solution of Eq. (1) still exists but it is no longer homogeneous.

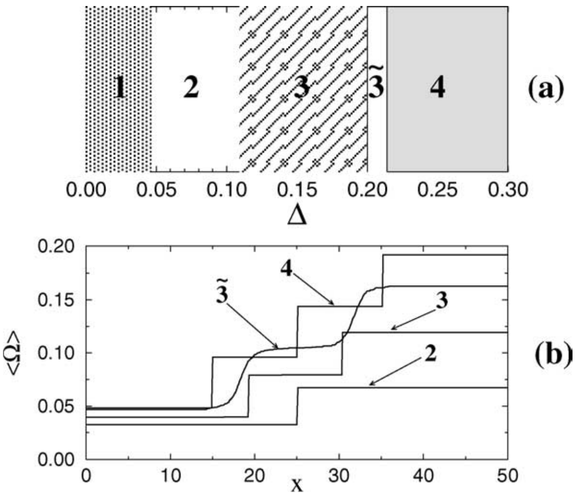

Fig. 1. A fragment of one-parameter diagram of the regimes in the system (1) at $g = 0.9$ (a) and the corresponding structures of frequency clusters (b). Numerals mark the numbers of the clusters. Region $\tilde{3}$ relates to the intermediate structure with three clusters.

This means that all points of the system continue to oscillate with equal frequencies but not in-phase. This is the regime of global frequency synchronization. With further increasing of Δ the regime of global frequency synchronization breaks and is transformed into the regime of partial (cluster) synchronization. The number of clusters depends on the value of Δ (at a given fixed diffusion coefficient g) [9]. Similarly to the case of coupled self-sustained oscillators [7] the observed structures of clusters can be divided into "perfect" and "imperfect" ("intermediate"). In the case of intermediate clusters the frequency inside a cluster slightly varies and there are no precise boundaries between them (see case $\tilde{3}$ in Fig. 1(b)). For the perfect structures the average frequency Ω inside a cluster remains constant, and the boundaries between clusters are sharp (cases 1, 2, 3 and 4). In Fig. 1(a) we built regions of the mismatch parameter Δ with qualitatively different structures of clusters. The less mismatch corresponds to a smaller number of clusters. The small mismatch leads to the global synchronization when all points of the system oscillate with the same frequency (region 1 in Fig. 1(a)) that is approximately equal to $\Delta/2$. With increasing frequency gradient Δ/l, from $\Delta \simeq 0.045$ we observe a sudden rebuilding of the spatial structure resulting in the appearance of two frequency clusters (region 2). Then, from

Δ being approximately equal to 0.11 the spatial structure is transformed into three clusters (region 3). At $\Delta \simeq 0.2$ the three-cluster structure becomes intermediate and gradually changes to the perfect four-cluster structure (region 4). With further increasing of the frequency gradient more and more short cluster structures appear successively. Regions of two consequent structures are divided by the region with an intermediate cluster structure. Inside each region marked in Fig. 1 the frequency of each cluster and their location can vary but their number remains constant.

Now let us consider the two coupled systems (2) in the regime of frequency clusters. We explore the distribution of the average frequencies $\Omega_{1,2}$ along spatial coordinates in both systems depending on the mismatch value $\delta = \Delta_1 - \Delta_2$ and the coupling parameter ε. The lengths of both systems were chosen as $l = 50$ and the diffusion coefficients $g_1 = g_2 = 0.9$. We fix the parameter $\Delta_1 = 0.16$ and change the values of Δ_2 and coupling ε. At $\varepsilon = 0$ in the neighborhood of $\Delta_2 = \Delta_1$, both systems demonstrate spatial structures with three frequency clusters. However, if $\Delta_1 \neq \Delta_2$, the average frequencies in two systems do not coincide. For a larger value of the mismatch parameter δ the number of clusters in the second medium becomes $N = 2$ or $N = 4$ depending on the sign of δ.

Let us choose the value of the mismatch $\delta = -0.08$ related to the structure of different numbers of perfect clusters in both systems when they are uncoupled (Fig. 2(a)). At $\varepsilon \neq 0$ the elements of the systems tend to synchronize their oscillations at equal frequencies. With increasing ε the synchronization begins from the first boundary of the systems (at small x) since in this case the difference between natural frequencies $\omega_1 - \omega_2$ is minimal. This first step of synchronization leads to the formation of intermediate cluster structures. Then the process of synchronization involves more and more elements of the systems. While coupling is small the number of clusters in both systems remains different (Fig. 2(b), (c)). Starting with a certain value of the coupling strength dependences of average frequencies on the spatial coordinate $\Omega_{1,2}(x)$ become completely identical (Fig. 2(d)). The resulted synchronized cluster structures are similar to those observed at $\varepsilon = 0$, $\Delta = 0$ when the both media contains three clusters.

Fig. 3 demonstrates a fragment of the diagram of the regimes for two coupled systems (2) on the δ–ε plane. There we built two regions. The first one relates to complete cluster synchronization 3–3 (it is marked by grey colour) when the spatial synchronization of cluster structures in both systems is accompanied by exact coincidence of frequencies in each point of the space $\Omega_1(x) \equiv \Omega_2(x)$. In this case the synchronization of spatial structures is a direct con-

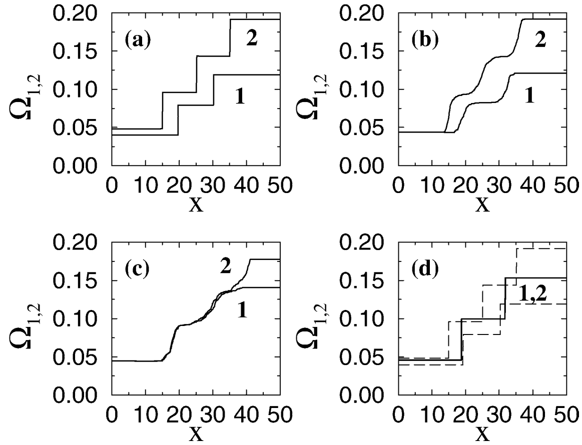

Fig. 2. Dependences of average frequencies $\Omega_{1,2}$ on the spatial coordinate x for fixed $\delta = -0.08$ and different values of the coupling strength: $\varepsilon = 0$ (a), $\varepsilon = 0.005$ (b), $\varepsilon = 0.03$ (c), $\varepsilon = 0.045$ (d). Curves 1 and 2 correspond to the first and second medium. Dashed lines in (d) repeat the distributions of frequencies for uncoupled media from (a).

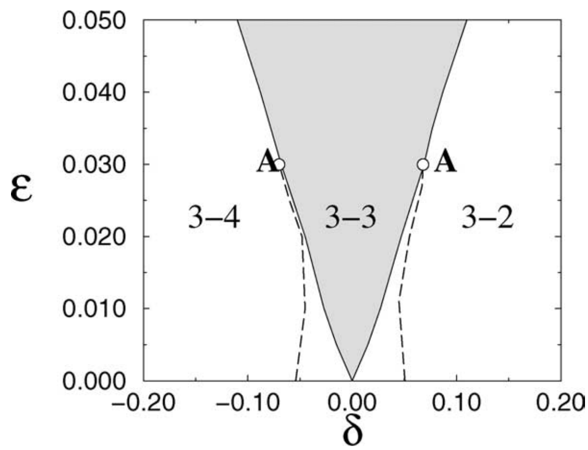

Fig. 3. Region of clusters synchronization on δ–ε parameter plane (at $\Delta_1 = 0.16$, $g_{1,2} = 0.9$). Grey colour selects the region of frequency synchronization between subsystems in each point of the media. Dashed lines mark the area with the equal numbers of clusters in both media. Figures denote the numbers of clusters in the first and second medium.

sequence of the classical phenomenon of "frequency locking" which takes place in each point of the self-sustained oscillating systems. However, contrary to the case of synchronization of oscillators, when the frequency of synchronized oscillators is located between frequencies of uncoupled ones, in the case of interacting media the synchronization frequency can be above them (this can be seen from Fig. 2(d)). Generally, the frequency of inter-cluster synchronization in the every point of space $\Omega(x)$ depends not only on frequencies in the media $\Omega_{1,2}(x)$ but also on frequencies in the neighbouring points $\Omega_{1,2}(x \pm dx)$. The region of synchronization has a typical "tongue-like" shape which is a characteristic form of areas for classical synchronization [15]. Besides, there is a region (bounded by dash lines) where both systems have the same numbers of clusters (three in the every medium), but their frequencies do not coincide. The clusters in this region are imperfect (Fig. 2(b), (c)). Increasing the coupling strength ε leads to the widening of the synchronization. For a rather large coupling (over the points A) there exists only the region of complete FS of the two systems. The change of the mismatch parameter δ from outside of the synchronization region into inside leads to a sharp rebuilding of the cluster structure from "3–4" to "3–3" clusters, which is accompanied by full frequency locking in each point of the space.

In conclusion, we showed that the mechanism of the FS of spatial structures is based on the phenomenon of frequency locking in each point of the system. This mechanism is a new manifestation of the fundamental synchronization phenomena.

Acknowledgements

The work is supported by the Civilian Research and Development Foundation of the US (grant SR-006-01). T.V. also thanks INTAS for partial financial support (grant 01-2061). T.K. acknowledges the support of KBN (Poland) under Grant No. 7TO7A 005 19. A.A. and V.A. acknowledge the hospitality of the Technical University of Lodz.

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