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Evaluation of the largest Lyapunov exponent in dynamical systems with time delay

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Abstract

The method of estimation of the largest Lyapunov exponents for dynamical systems with time delay has been developed. This method can be applied both for flows and discrete maps. Our approach is based on the phenomenon of synchronization of identical systems coupled by linear negative feedback mechanism (flows) and exponential perturbation (maps). The existence of linear dependence of the largest Lyapunov exponent on the coupled parameter allows the precise estimation of this exponent.

1. Introduction

The idea of Lyapunov exponents is to define characteristic numbers for a dynamical system that allow to classify the behaviour of the system in a concise manner. These numbers should account for exponential convergence or divergence of trajectories that start close to each other. The existence of such numbers has been proved by Oseledec theorem [13]. The number of Lyapunov exponents, which characterize the behaviour of dynamical system, is equal to the dimension of this system. If the sum of all Lyapunov exponents is negative than the system has an attractor. For practical applications it is most important to know the largest Lyapunov exponent (LLE in further notation). If the largest value in the spectrum of Lyapunov exponents is positive, it means that the system is chaotic. The largest value equal to zero indicates periodic or quasi-periodic dynamics. If all Lyapunov exponents are negative then the stable critical point is an attractor.

The first numerical study of the system behaviour using Lyapunov exponents has been done by Henon and Heiles [8], before the publication of Oseledec theorem. The first robust algorithms for calculating Lyapunov exponents for the system given by a continuous and differentiable ordinary differential equation have been developed by Benettin et al. [2] and Shimada and Nagashima [19], and later improved by Benettin et al. [3,4] and Wolf [25]. Such algorithms allow an easy estimation of entire spectrum of Lyapunov exponents but they do not work for the system with discontinuities or time delay and in the case when the equations of motion describing dynamics of the system are unknown. In such a case the estimation of Lyapunov exponents is not straightforward [9,11]. One possible approach is the estimation of Lyapunov

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exponents from the scalar time series basing on Takens procedure [23]. Numerical algorithms for such estimation have been developed by Wolf et al. [24], Sano and Sawada [18] and later improved by Eckmann et al. [5], Rosenstein et al. [17] and Parlitz [14].

In this paper we propose the other method of estimation of the largest Lyapunov exponent of the system with time delay. Our method is based on the phenomenon of synchronization of coupled identical systems and it does not use directly Oseledec theorem. Previously we successfully used the similar method for dynamical systems with discontinuities.

2. Method

Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ denote typical phase space trajectories of two identical dynamical systems. The aim of our method is to estimate the largest Lyapunov exponent of these systems. The method is based on the phenomenon of full ($\lim_{t\to\infty}\|\mathbf{x}(t)-\mathbf{y}(t)\| = 0$) [15] or practical ($\lim_{t\to\infty}\|\mathbf{x}(t)-\mathbf{y}(t)\| < \delta$, where δ—is a small parameter) [10] synchronization of identical dynamical systems. As the full description of this method can be found in [6,20–22] here we only give the brief description of it.

Consider two identical dynamical systems $\mathbf{x} = \mathbf{f}(\mathbf{x})$ and $\mathbf{y} = \mathbf{f}(\mathbf{y})$, which are unidirectionally coupled by the negative feedback mechanism

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{D}(\mathbf{y} - \mathbf{x}), \tag{1a}$$

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}), \tag{1b}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ and $\mathbf{D} = \mathrm{diag}[d, d, \ldots, d] \in \mathbb{R}^k$ is the diagonal matrix of the coupling, $d \in \mathbb{R}$ represents a coupling coefficient. Such systems fully synchronize when

$$d > \lambda_1, \tag{2}$$

where λ_1 is the largest Lyapunov exponent [6,20,21]. This linear dependence between the rate of coupling and the LLE results from the linearized solution of Eqs. (1a) and (1b), describing the time evolution of a distance $\mathbf{z}(t) = \mathbf{x}(t) - \mathbf{y}(t)$ between the trajectories representing both coupled systems, for small norm of vector $\mathbf{z}(t)$. Such a solution (starting from $\mathbf{z}(0)$) has a form:

$$\|\mathbf{z}(t)\| = \|\mathbf{z}(0)\| \exp(\lambda_1 - d)t. \tag{3}$$

The same synchronization condition (inequality (2)) showing the linear dependence of λ_1 and d can be obtained for the pair of identical maps, where one of them $\mathbf{y}_{n+1} = \mathbf{f}(\mathbf{y}_n)$ perturbs the second one $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$ with the exponential signal $\Delta\mathbf{y}_n = (\mathbf{x}_n - \mathbf{y}_n)\exp(-d)$. This perturbation models the effect of the linear negative feedback mechanism as follows:

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{y}_n + \Delta\mathbf{y}_n), \tag{4a}$$

$$\mathbf{y}_{n+1} = \mathbf{f}(\mathbf{y}_n), \tag{4b}$$

$$\Delta\mathbf{y}_{n+1} = [\mathbf{f}(\mathbf{y}_n + \Delta\mathbf{y}_n) - \mathbf{f}(\mathbf{y}_n)]\exp(-d), \tag{4c}$$

where $\mathbf{x}$, $\mathbf{y}$, $\Delta\mathbf{y} \in \mathbb{R}^k$. When the synchronization condition (inequality (2)) is fulfilled $\mathbf{x} = \mathbf{y}$ and $\Delta\mathbf{y} = \mathbf{0}$ [22].

In numerical calculations the value of coupling or perturbation coefficient d for which synchronization is achieved can be obtained as follows:

1. from the integration of augmented systems (Eqs. (1a) and (1b))—for the flows (*case 1* in further considerations),
2. from the iterations of coupled maps (Eqs. (4a), (4b) and (4c)—for discrete systems (*case 2*),
3. from the iterations of coupled maps (Eqs. (4a), (4b) and (4c), where maps were reconstructed from the flow given by Eq. (1a) (*case 3*).

Additionally, the effectiveness of numerical estimation of LLE can be improved by the application of the procedures described in [20,22].

The *case 1* allows direct estimation of the LLE λ_1, in the *case 3* one directly estimates exponent λ_m, where

$$\lambda_m = \lambda_p T_A, \tag{5}$$

and T_A is the average time between successive points of the map.

3. Application to the systems with time delay

Dynamical systems with time delay are described either by the retarded differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-\tau)), \tag{6}$$

in the case of flows or by the map

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, \mathbf{x}_{n-m}), \tag{7}$$

in the case of discrete systems, where $\mathbf{x} \in \mathbb{R}^k$, $\tau \in \mathbb{R}$ and $m \in \mathbb{N}$ denote delay. Eqs. (6) and (7) can be easily integrated numerically but due to the retarded arguments in ($\mathbf{x}(t-\tau)$ or $\mathbf{x}_{n-m}$), Lyapunov exponents cannot be calculated using algorithms based on the Oseledec theorem.

In the case of Eq. (6) one can develop argument $\mathbf{x}(t-\tau)$ into the Taylor's series in the neighbourhood of τ

$$\mathbf{x}(t-\tau) = \mathbf{x}(t) + \frac{1}{i!}\sum_i (-\tau)^i \frac{\mathrm{d}^i\mathbf{x}(t)}{\mathrm{d}t^i}, \tag{8}$$

where $i \in \mathbb{N}$. After this transformation the retarded differential equation (6) becomes ordinary differential equation with delay τ as one of the constant parameters. This transformation gives some approximation of the dynamics governed by Eq. (6) only in the case of small time delay τ for which it is sufficient to consider only a few first components of power series (8). It should be noted here that the dynamics governed by Eq. (6) is very sensitive to the changes of the delay τ, particularly in the case of chaotic behaviour, so approximation Eq. (8) rarely gives good results.

Description presented in Section 2 showed that coupling should be applied to all state variables (diagonal coupling). In dynamical systems without time delay this has an effect in the next iteration. In the case of retarded dynamical systems the effect of coupling is visible after the time defined by parameters τ or m. This causes that synchronization of differential equations (1a) and (1b) or maps (4a) and (4b) is obtained after more iterations but inequality (2) still holds. In practice, our synchronization based method of estimation of the LLE for systems with time delay can be implemented by substitution of (6) or (7) appropriately to Eqs. (1a) and (1b) or (4a), (4b) and (4c).

4. Numerical examples

As the numerical examples we consider estimation of LLE λ_1 for the following dynamical systems with time delay:

1. Henon map as the example of discrete system.
2. Van der Pol oscillator as the example of the flow.

4.1. Henon map with time delay

Consider Henon map [7] with delay which is given as follows:

$$\begin{aligned} y(1)_{n+1} &= 1 - ay(1)_n^2 + y(2)_{n-m}, \\ y(2)_{n+1} &= by(1)_n, \end{aligned} \tag{9}$$

where argument $y(2)$ is delayed by m iterations. Estimation of the LLE of the system (9) can be performed by the iteration of the augmented system:

$$\begin{aligned} x_1^* &= 1 - a[y_1 + \Delta y_1]^2 + [y_{2d} + \Delta y_{2d}], \\ x_2^* &= b[y_1 + \Delta y_1], \\ y_1^* &= 1 - ay_1^2 + y_{2d}, \\ y_2^* &= by_1, \\ \Delta y_1^* &= [\Delta y_{2d} - a\Delta y_1(\Delta y_1 + 2y_1)]\exp(-d), \\ \Delta y_2^* &= b\Delta y_1 \exp(-d). \end{aligned} \tag{10}$$

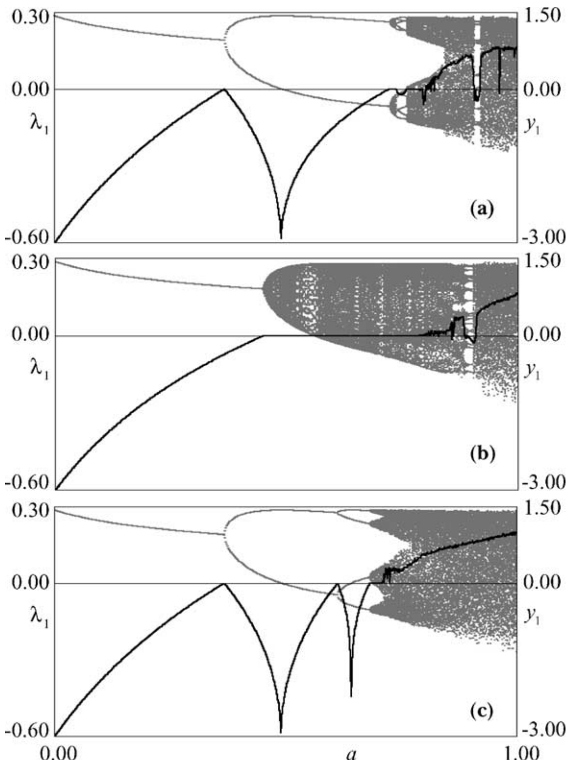

Fig. 1. Bifurcation diagrams of the delayed Henon map (9) (in grey) and LLEs (in black); $b = 0.3$: (a) $m = 4$, (b) $m = 6$, (c) $m = 10$.

In order to simplify the description of k-dimensional maps $\mathbf{y}_{n+1} = \mathbf{f}(\mathbf{y}_n)$, in Eq. (10) we denoted: $y_i^* = y(i)_{n+1}$, $y_i = y(i)_n$ and $y_{id} = y(i)_{n-m}$. In numerical calculations we took $b = 0.3$, different values of delay $m = 4$, $m = 6$, $m = 10$ and considered a as a control parameter. The estimation has been performed according to the *case 2* (as described in Section 2).

In Fig. 1(a–c) we presented the results of our estimation of LLE together with bifurcation diagrams for different values of the delay (Fig. 1(a) $m = 4$, Fig. 1(b) $m = 6$, Fig. 2(c) $m = 10$). The positive values of LLE λ_1 coincides with the regions of chaotic behaviour detected from the bifurcation diagrams.

Fig. 2(a and b) shows the estimated values of LLE λ_1 in the two-dimensional parameter space (a, b) of the system (9).

4.2. Van der Pol oscillator with time delay

Consider nonlinear Van der Pol oscillator with external harmonic excitation and time delay [1,16]

$$\ddot{x} - \alpha(1 - x^2)\dot{x} + \beta x^3 = \kappa x(t - \tau) - p\cos(\eta t), \tag{11}$$

where α, β, p, η, κ and time delay τ are constant.

For the Eq. (11) the estimation of the LLE can be performed as in the *case 1* and *3*, as described in Section 2.

In the *case 1* of the estimation procedure, after the change of variables ($x = x_1$, $\dot{x} = x_2$) and substitution of Eq. (11) into Eqs. (1a) and (1b) we obtain the augmented system

$$\begin{aligned}
\dot{x}_1 &= x_2 + d(y_1 - x_1),\\
\dot{x}_2 &= \kappa x_1(t - \tau) - p\cos(\eta t) + \alpha(1 - x_1^2)x_2 - \beta x_1^3 + d(y_2 - x_2),\\
\dot{y}_1 &= y_2,\\
\dot{y}_2 &= \kappa y_1(t - \tau) - p\cos(\eta t) + \alpha(1 - y_1^2)y_2 - \beta y_1^3,
\end{aligned} \tag{12}$$

which is numerically integrated.

To implement the *case 3* of the estimation procedure one can use a Poincare map, but it should be noted that it requires the special selection of the integration step Δt (both time delay τ and period of excitation $T = 2\pi/\eta$ should

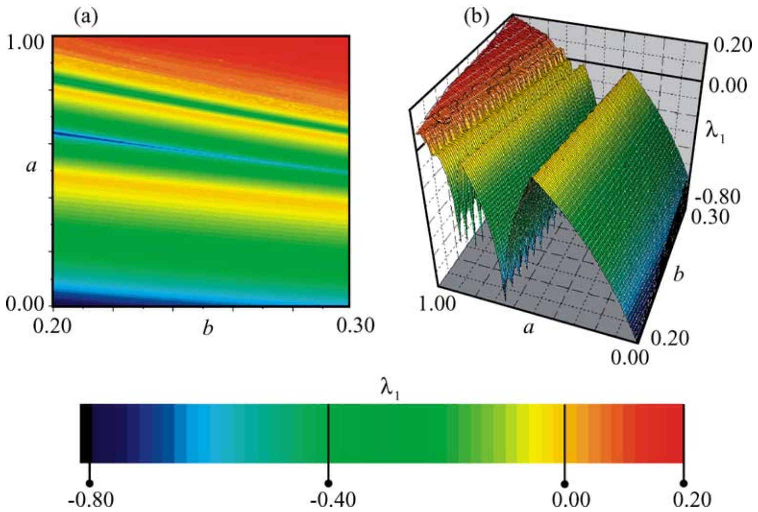

Fig. 2. LLE of the delayed Henon map (9); $m = 10$, (a) projection into (a,b) plane, (b) three-dimensional view.

be natural multiplication of Δt). For the nonautonomous systems with time delay as Eq. (11) the alternative approach based on the so called τ-map (Σ_τ) is possible. Such a map is constructed from the iterations of the flow after the successive periods of time equal to time delay τ. The τ-map of the system (11) is given by

$$\begin{aligned} y_{n+1} &= f(y(t)_n, y(t-\tau)_n, \dot{y}(t)_n), \\ \dot{y}_{n+1} &= g(y(t)_n, y(t-\tau)_n, \dot{y}(t)_n), \end{aligned} \tag{13}$$

where f and g are unknown analytically. However, the τ-map can be numerically obtained from the flow of the system (11) according to the following relation defining transition from n to $n + 1$ iteration

$$\sum_\tau = \begin{cases} y_n = y(t_n) \to y_{n+1} = y(t_n + \tau), \\ \dot{y}_n = \dot{y}(t_n) \to \dot{y}_{n+1} = \dot{y}(t_n + \tau). \end{cases} \tag{14}$$

The assumption that time interval between the successive iterations is equal to τ causes that the τ-map takes the form of the map (7) with delay $m = 1$, so $y(t-\tau)_n = y_{n-1}$. This allows us to transform Eqs. (4a–c) into the six-dimensional system

$$\begin{aligned} x_{n+1} &= f(y_n + \Delta y, y_{n-1} + \Delta y_{n-1}, \dot{y}_n + \Delta\dot{y}), \\ \dot{x}_{n+1} &= g(y_n + \Delta y, y_{n-1} + \Delta y_{n-1}, \dot{y}_n + \Delta\dot{y}). \\ y_{n+1} &= f(y_n, y_{n-1}, \dot{y}_n), \\ \dot{y}_{n+1} &= g(y_n, y_{n-1}, \dot{y}_n), \\ \Delta y_{n+1} &= [f(y_n + \Delta y, y_{n-1} + \Delta y_{n-1}, \dot{y}_n + \Delta\dot{y}) - f(y_n, y_{n-1}, \dot{y}_n)]\exp(-d), \\ \Delta\dot{y}_{n+1} &= [g(y_n + \Delta y, y_{n-1} + \Delta y_{n-1}, \dot{y}_n + \Delta\dot{y}) - g(y_n, y_{n-1}, \dot{y}_n)]\exp(-d), \end{aligned} \tag{15}$$

which numerically implements the *case 3* of the estimation procedure. From Eq. (5) one gets the following relation between the LLEs λ_1 and exponent $\lambda_{1\tau}$ estimated using τ–maps in the *case 3* of the estimation procedure

$$\lambda_1 = \frac{\lambda_{1\tau}}{\tau}. \tag{16}$$

In the numerical calculation we took $\alpha = 0.20$, $\beta = 1.00$, $\tau = 2.00$, $\eta = 4.00$, $p = 17.00$ and assumed κ as a control parameter. The examples of τ-maps of the system (13) and the estimated values of the LLE $\lambda_{1\tau}$ (estimated according to the *case 3* of the Section 2) are shown in Fig. 3(a–c). Noncommensure ratio of the excitation period T and delay τ causes that τ-maps look similar to the phase portraits for sufficiently large number of iteration. These results are compared with the estimation results obtained using the *case 1* of the Section 2. The appropriate Poincare maps and estimated values of LLE are shown in Fig. 3(d–f). Fig. 4(a–c) shows the comparison of the bifurcation diagram of the system (11) (Fig. 4(a)) with the bifurcation diagram of LLE obtained in the *case 1* (Fig. 4(b) and *case 3* (Fig. 4(c)). Good accuracy of the results obtained in both cases is clearly visible (Eq. (16) is fulfilled).

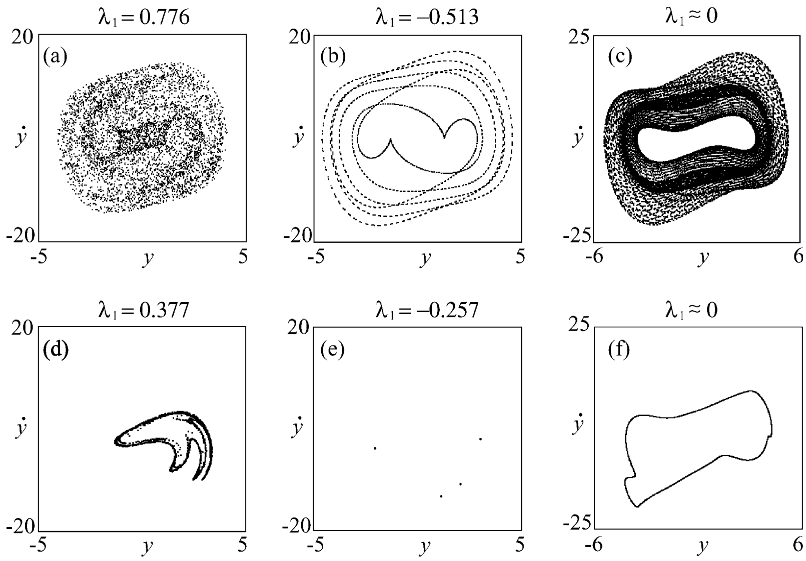

Fig. 3. The τ-maps (a–c) and equivalent Poincare maps (d–f) of the delayed Van der Pol oscillator (11) together with estimated values of the LLE λ_1; $\alpha = 0.20$, $\beta = 1.00$, $\tau = 2.00$, $\eta = 4.00$, $p = 17.00$: (a, d) chaotic behaviour $\kappa = 0.80$, (b, e) periodic behaviour $\kappa = 2.20$, (c, f) quasi-periodic behaviour $\kappa = 6.00$. In (a–c) the *case 3* and in (d–f) the *case 1* has been used.

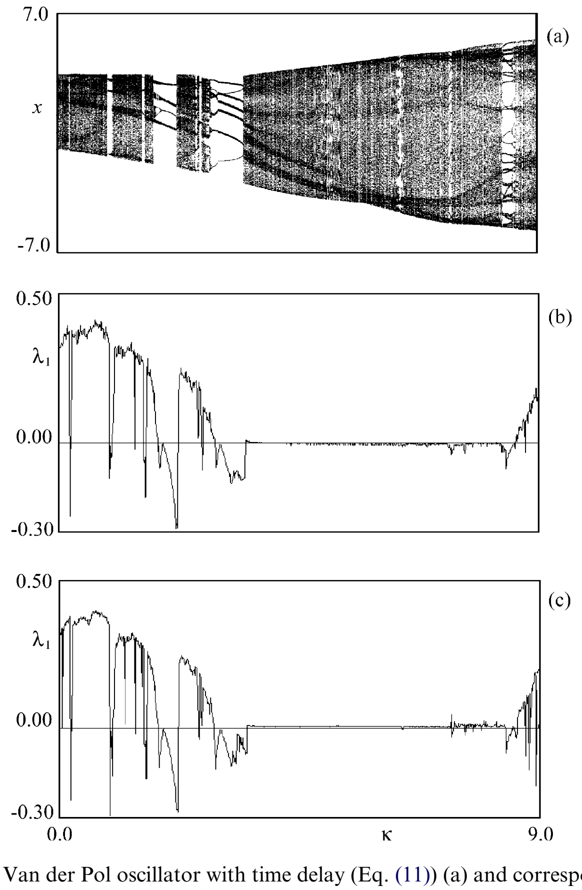

Fig. 4. Bifurcation diagram of the Van der Pol oscillator with time delay (Eq. (11)) (a) and corresponding LLE estimated by means of *case 1* (b) and *case 3* (c) of the method. $\alpha = 0.20$, $\beta = 1.00$, $\tau = 2.00$, $\eta = 4.00$, $p = 17.00$.

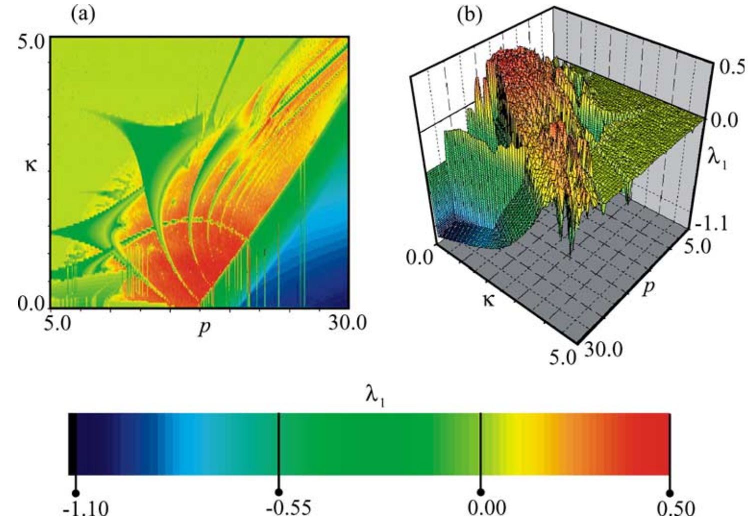

Fig. 5. LLE of the Van der Pol oscillator with time delay (Eq. (11)); projection into plane (a) and three-dimensional view (b), in κ and p parameters space; $\alpha = 0.20$, $\beta = 1.00$, $\tau = 2.00$, $\eta = 4.0$. The *case 1* of the method has been used.

In Fig. 5(a and b) the estimated value of the LLE λ_1 of the system (11) in the two-dimensional parameter space (κ, p) is presented. Note the fractal structure of the $\lambda_1(\kappa, p)$ manifold.

As it was mentioned in Section 3, for small values of time delay the retarded differential equation (11) can be approximated by appropriate ordinary differential equation. Substituting the Taylor series

$$x(t-\tau) = x - \tau\dot{x} + \frac{1}{2}\tau^2\ddot{x} - \frac{1}{6}\tau^3\dddot{x} + \cdots \tag{17}$$

into Eq. (11) and considering only its first three components one gets

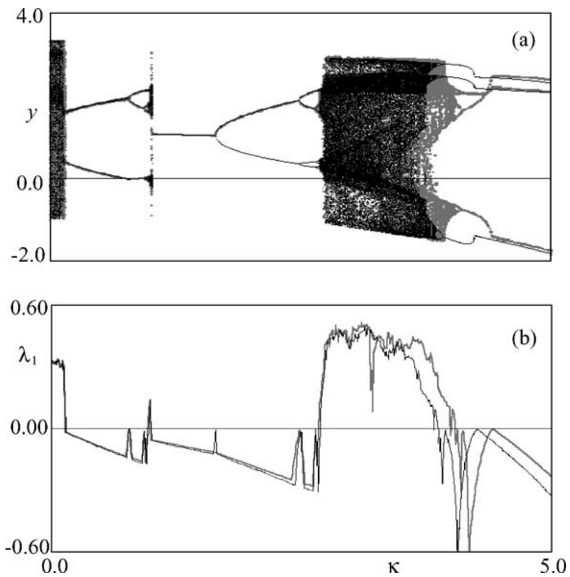

Fig. 6. (a) The comparison of the bifurcation diagrams of the Van der Pol oscillator with time delay obtained from Eqs. (18) (in black) and Eq. (11) (in grey). (b) The corresponding LLE calculated by means of classical algorithm (black line) and estimated using *case 1* of the proposed method (grey line); $\alpha = 0.20$, $\beta = 1.00$, $\tau = 0.20$, $\eta = 4.00$, $p = 17.00$.

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = \frac{1}{1 - \frac{1}{2}\kappa\tau^2}[-p\cos(\eta t) + (\alpha x(1 - x_1^2) - \kappa\tau)x_2 - \beta x_1^3 + \kappa x_1]. \tag{18}$$

Eq. (18) allows direct calculation of Lyapunov exponents from algorithms based on the Oseledec theorem [2–4,19,25].

The comparison of this approximation with the results obtained by our synchronization based method is shown in Fig. 6 (numerical calculations have been performed for $\tau = 0.2$). In Fig. 6(a) we showed the comparison of bifurcation diagrams of Eq. (11) (in grey) and Eq. (18) (in black). In Fig. 6(b) one can find the comparison of the corresponding values of LLE estimated from Eq. (11) by synchronization method and calculated from Eq. (18) using software DYNAMICS [12]. For the small values of κ (small impact of the retarded argument on the dynamics of the system—Eq. (11)) the results obtained by both methods are similar but for larger values of κ these results diverge from each other.

5. Conclusions

In summary, we conclude that the proposed method allows the estimation of LLE for the dynamical systems with time delay. Our numerical examples showed that synchronization based approach can be applied for such systems no matter how large the delay is. The novelty of the method lies in its use of the synchronization phenomenon (readily recognized). Since the synchronization is easily detectable, the method has significant practical advantage over more traditional algorithmic methods, especially in dealing with systems with delay and also in dealing with nonsmooth systems.

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References

[1] Awrejcewicz J, Wojewoda J. Observation of chaos in a nonlinear oscillator with delay: numerical study. KSME J 1989;3(1):15–24.
[2] Benettin G, Galgani L, Strelcyn JM. Kolmogorov entropy and numerical experiment. Phys Rev A 1976;14:2338–45.
[3] Benettin G, Galgani L, Giorgilli A, Strelcyn JM. Lyapunov exponents for smooth dynamical systems and Hamiltonian systems; a method for computing all of them, part I: theory. Meccanica 1980;15:9–20.
[4] Benettin G, Galgani L, Giorgilli A, Strelcyn JM. Lyapunov exponents for smooth dynamical systems and Hamiltonian systems; a method for computing all of them, part II: numerical application. Meccanica 1980;15:21–30.
[5] Eckmann JP, Kamphorst SO, Ruelle D, Ciliberto S. Lyapunov exponents from a time series. Phys Rev Lett 1986;34(9):4971–9.
[6] Fujisaka H, Yamada T. Stability theory of synchronized motion in coupled-oscillator systems. Prog Theor Phys 1983;69(1):32–47.
[7] Henon M. A two dimensional map with a strange attractor. Commun Math Phys 1976;50:69.
[8] Henon M, Heiles C. The applicability of the third integral of the motion: some numerical results. Astron J 1964;69:77.
[9] Hinrichs N, Oestreich M, Popp K. Dynamics of oscillators with impact and friction. Chaos, Solitons & Fractals 1997;4(8):535–58.
[10] Kapitaniak T, Sekieta M, Ogorzalek. Monotone synchronization of chaos. Bifurcat Chaos 1996;6:211–5.
[11] Müller P. Calculation of Lyapunov exponents for dynamical systems with discontinuities. Chaos, Solitons & Fractals 1995;5(9):1671–81.
[12] Nusse HE, Yorke JA. Dynamics: numerical explorations. New York: Springer-Verlag; 1997.
[13] Oseledec VI. A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. Trans Moscow Math Soc 1968;19:197–231.
[14] Parlitz U. Identification of true and spurious Lyapunov exponents from time series. J Bifurcat Chaos 1992;2(1):155–65.
[15] Pecora LM, Carroll TL. Synchronization of chaos. Phys Rev Lett 1990;64:221–4.
[16] Plauth RH, Hsieh J-C. Chaos in a mechanism with time delays under parametric and external excitation. J Sound Vibrat 1987;114(1):73–90.
[17] Rosenstein MT, Collins JJ, De Luca CJ. A practical method for calculating largest Lyapunov exponents from small data sets. Physica D 1993;65(1,2):117–34.
[18] Sano M, Sawada Y. Measurement of the Lyapunov spectrum from a chaotic time series. Phys Rev Lett 1985;55:1082–5.

[19] Shimada I, Nagashima T. A numerical approach to ergodic problem of dissipative dynamical systems. Prog Theor Phys 1979;61(6):1605–16.

[20] Stefanski A, Kapitaniak T. Using chaos synchronization to estimate the largest Lyapunov exponent of non-smooth systems. Discrete Dyn Nat Soc 2000;4:207–15.

[21] Stefanski A. Estimation of the largest Lyapunov exponent in systems with impacts. Chaos, Solitons & Fractals 2000;11(15):2443–51.

[22] Stefanski A, Kapitaniak T. Estimation of the dominant Lyapunov exponent of non-smooth systems on the basis of maps synchronization. Chaos, Solitons & Fractals 2003;15:233–44.

[23] Takens F. Detecting strange attractors in turbulence. Lect Notes Math 1981;898:366.

[24] Wolf A, Swift JB, Swinney HL, Vastano JA. Determining Lyapunov exponents from a time series. Physica D 1985;16:285–317.

[25] Wolf A. Quantifying chaos with Lyapunov exponents. In: Holden V, editor. Chaos. Manchester: Manchester University Press; 1986. p. 273–90.