AIP Chaos
An Interdisciplinary Journal of Nonlinear Science
25TH ANNIVERSARY

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Citation: Chaos **22**, 047503 (2012); doi: 10.1063/1.4740460
View online: http://dx.doi.org/10.1063/1.4740460
View Table of Contents: http://scitation.aip.org/content/aip/journal/chaos/22/4?ver=pdfcov
Published by the AIP Publishing

Synchronous rotation of the set of double pendula: Experimental observations

J. Strzalko,[1] J. Grabski,[1] J. Wojewoda,[1] M. Wiercigroch,[2] and T. Kapitaniak[1]
[1]*Division of Dynamics, Technical University of Lodz, Stefanowskiego 1/15, 90-924 Lodz, Poland*
[2]*Centre for Applied Dynamics Research, School of Engineering, University of Aberdeen, AB24 3UE Aberdeen, Scotland*

(Received 26 March 2012; accepted 12 July 2012; published online 14 December 2012)

We study the occurrence of the synchronous rotation of a set of four uncoupled nonidentical double pendula arranged into a cross structure mounted on a vertically excited platform. Under the excitation, the pendula can rotate in different directions (counter-clockwise or clockwise). It has been shown that after a transient, many different types of synchronous configurations with the constant phase difference between pendula can be observed. The experimental results qualitatively agree with the numerical simulations. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4740460]

Recent investigations have shown that the large sets of oscillators (either coupled self-excited oscillators or uncoupled passive oscillators externally excited by the common signal) have a great potential in a large amount of application areas ranging from physics and engineering to economy and biology. The main interest in these studies is focussed on the occurrence of the synchronous behavior. This phenomenon offers the most fundamental example of emergent behavior. Synchronous states are widely observed in nature and are common in mechanical oscillatory systems. We study the dynamics of four nonidentical parametrically externally excited double pendula. Our goal is to identify the possible synchronous configurations.

NOMENCLATURE

A = Amplitude of the parametric excitation
ω = Frequency of the parametric excitation
G = Earth gravity
ξ_{i1}, ξ_{i2} = Coordinates of pendulums' centres in their local systems
k_{is} = Spring stiffness coefficient
k_{ic} = Damping coefficient
$m_{i1}, J_{i1}, m_{i2}, J_{i2}$ = Masses and moments of inertia of the I_i and II_i pendula
$\varphi_{i1}, \varphi_{i2}$ = Angles of pendula position
T, V = Kinetic and potential energies of the system

I. INTRODUCTION

A parametrically excited pendulum is an archetype for strongly nonlinear dynamical systems, which naturally has been given a great deal of attention in literature (for example, Refs. 1–12). Such a pendulum can perform both oscillatory and rotational motions. Usually, more attention has been paid to the oscillatory motion but recently more and more studies are concentrated on the rotational motion.[13–18] The rotational responses of the pendulum attracted more interest due to the concept of extracting energy from sea waves using pendulum dynamics proposed by Wiercigroch.[19–21] Such a pendulum system can be used for converting pendulum base oscillations into the rotational motion of the pendulum mass (the oscillations of the base are caused by the sea waves, whereas the pendulum rotational motion provides a driving torque for an electrical generator).

Mechanical systems that contain rotating parts (for example, vibro-exciters, unbalance rotors) are typical in engineering applications and for years have been the subject of intensive studies.[22–24] One problem of scientific interest, which among others occurs in such systems, is the phenomenon of synchronization of different rotating parts (Refs. 25 and 26, and references within). Despite different initial conditions, after a sufficiently long transient, the rotating parts move in the same way—complete synchronization, or a permanent constant shift is established between their displacements, i.e., the angles of rotation—phase synchronization (Refs. 27–34). Synchronization occurs due to dependence of the periods of rotating elements motion and the displacement of the base on which these elements are mounted.[35]

In the previous paper,[36] we consider the dynamics of the system consisting of n pendula mounted on the movable beam. The pendula are excited by the external torques which are linearly dependent on the angular velocities of the pendulums. As the result of such excitation, each pendulum rotates around its axis of rotation. It has been shown that both complete and phase synchronizations of the rotating pendulums are possible. We have derived the approximate analytical conditions for both types of synchronizations and equations which allow the estimation of the phase differences between the pendulums. Contrary to the case of the oscillatory pendulums,[38–40] phase synchronization is not limited to three and five clusters' configurations. We have considered the case of slowly rotating pendulums and the influence of the gravity on their motion. Our results have been compared to those of Blekhman.[26]

The dynamics of the similar system in which one pendulum rotates counter-clockwise, i.e., has a positive angular velocity, while the remaining pendula rotate clockwise with negative angular velocity has been studied in Ref. 37. We have considered two cases: (i) pendula rotate in the horizontal plane, i.e., the gravity has no influence on their motion, (ii) pendula rotate in the vertical plane and their weight of causes the unevenness of their rotation, i.e., each pendulum slows down when the center of its mass goes up and accelerates when the center of its mass goes down. We show that in such systems, despite opposite directions of rotation different types of synchronization occur.

In this paper, we consider the dynamics of the set of two pairs of double pendula mounted on the platform which oscillates vertically. Using a custom designed experimental rig, we identify different types of synchronous motion of rotating pendula, i.e., we observe the synchronization of pendula rotating both in the same and opposite directions. The rotating pendula can be 1:1 and 1:2 synchronized with the oscillations of the platform. The existence of experimentally observed synchronous states is confirmed in numerical simulations. We show the extreme sensibility of the synchronized state on the system parameters and initial conditions.

The paper is organized as follows. In Sec. II, we describe the considered model of the coupled rotating pendulums and the design of the experimental rig. The examples of the configurations of the synchronized pendulums are given in Sec. III. Finally, we summarize our results in Sec. IV.

II. THE MODEL

We consider the system of two pairs of double pendula arranged into a cross structure as shown in Figures 1(a)–1(c). The lower pendula's bobs (marked there with symbols II_1, II_2, II_3, II_4) can only rotate around their horizontal axes (at the points C_1, C_2, C_3, C_4. The upper bobs (I_1, I_2, I_3, I_4) can only oscillate around the horizontal axes marked by A_1, A_2, A_3, and A_4 and located on the base (*III*—in Figures 1(a) and 1(b)). The base, mounted on the shaker, is excited in the vertical direction by a kinematic excitation, $y = A\cos\omega t$.

To describe the dynamics of the system, we introduced the following generalized coordinates:

$$\mathbf{q} = [\varphi_{11}(t), \varphi_{12}(t), \varphi_{21}(t), \varphi_{22}(t), \varphi_{31}(t), \varphi_{32}(t), \varphi_{41}(t), \varphi_{42}(t)], \tag{1}$$

and generalized velocities

$$\dot{\mathbf{q}} = [\dot\varphi_{11}(t), \dot\varphi_{12}(t), \dot\varphi_{21}(t), \dot\varphi_{22}(t), \dot\varphi_{31}(t), \dot\varphi_{32}(t), \dot\varphi_{41}(t), \dot\varphi_{42}(t)]\,. \tag{2}$$

The position of the mass center of the ith upper pendula (I_i in Figure 1(c)) is given by

$$x_{i1} = \xi_{i1}\cos(\varphi_{i1}),\;\; y_{i1} = A\cos(\omega t) + \xi_{i1}\sin(\varphi_{i1}),\;\; i = 1\ldots 4 \tag{3}$$

and the position of mass center of the ith lower pendula (II_i in Figure 1(c)) by

$$\begin{aligned} x_{i2} &= \xi_{i1}\cos(\varphi_{i1}) - \xi_{i2}\sin(\varphi_{i2}),\\ y_{i2} &= A\cos(\omega t) + \xi_{i1}\sin(\varphi_{i1}) + \xi_{i2}\cos(\varphi_{i2}),\;\; i = 1\ldots 4. \end{aligned} \tag{4}$$

The velocities of the mass center of upper and lower pendula are read, respectively, as

$$\begin{aligned} v_{xi1} &= -\xi_{i1}\dot\varphi_{i1}\sin(\varphi_{i1}),\\ v_{yi1} &= -A\omega\sin(\omega t) + \xi_{i1}\dot\varphi_{i1}\cos(\varphi_{i1}),\;\; i = 1\ldots 4 \end{aligned} \tag{5}$$

and

$$\begin{aligned} v_{xi2} &= -\xi_{i1}\dot\varphi_{i1}\sin(\varphi_{i1}) - \xi_{i2}\dot\varphi_{i2}\cos(\varphi_{i2}),\;\; i = 1\ldots 4,\\ v_{yi2} &= -A\omega\sin(\omega t) + \xi_{i1}\dot\varphi_1\cos(\varphi_{i1}) - \xi_{i2}\dot\varphi_{i2}\sin(\varphi_{i2}),\\ & i = 1\ldots 4. \end{aligned} \tag{6}$$

Kinetic energy and potential energy of the whole system described as

$$\begin{aligned} 2T = \sum_{i=1}^{4}\Big[& m_{i1}((A\omega\sin(\omega t) - \xi_{i1}\cos(\varphi_{i1})\dot\varphi_{i1})^2\\ &+ \xi_{i1}^2\sin^2(\varphi_{i1})\dot\varphi_{i1}^2) + m_{i2}((\xi_{i2}\sin(\varphi_{i2})\dot\varphi_{i2}\\ &- \xi_{i1}\cos(\varphi_{i1})\dot\varphi_{i1} + A\omega\sin(\omega t))^2\\ &+ (\xi_{i1}\sin(\varphi_{i1})\dot\varphi_{i1} + \xi_{i2}\cos(\varphi_{i2})\dot\varphi_{i2})^2)\\ &+ J_{i1}\dot\varphi_{i1}^2 + J_{i2}\dot\varphi_{i2}^2\Big] \end{aligned} \tag{7}$$

and

$$\begin{aligned} V = \sum_{i=1}^{4}\Bigg[&\frac{1}{2}k_{is}\eta_{i1}^2\sin^2(\varphi_{i1}) - m_{i1}g(A\cos(\omega t) + \xi_{i1}\sin(\varphi_{i1}))\\ &- m_{i2}g(A\cos(\omega t) + \xi_{i1}\sin(\varphi_{i1}) + \xi_{i2}\cos(\varphi_{i2}))\Bigg]. \end{aligned} \tag{8}$$

Assuming the dissipated energy in the form of Rayleigh function,

$$R = \frac{1}{6}\sum_{i=1}^{4}k_{ic}\eta_{i2}^3\dot\varphi_{i2}^2, \tag{9}$$

one can derive the equations of motion of the system (using Lagrange's method)

$$\begin{aligned} &(J_{i2} + m_{i2}\xi_{i2}^2)\ddot\varphi_{i2} + m_{i2}\xi_{i2}\Big(A\omega^2\cos(\omega t) + g\Big)\sin(\varphi_{i2})\\ &\quad + m_{i2}\xi_{i2}\Big(\xi_{i1}\Big(\cos(\varphi_{i1} - \varphi_{i2})\dot\varphi_{i1}^2 + \sin(\varphi_{i1} - \varphi_{i2})\ddot\varphi_{i1}\Big)\Big)\\ &\quad + \frac{1}{3}k_{ic}\eta_{i2}^3\dot\varphi_{i2} = 0,\quad i = 1\ldots 4,\\ &\Big(J_{i1} + (m_{i1} + m_{i2})\xi_{i1}^2\Big)\ddot\varphi_{i1} + \frac{1}{2}\eta_{i1}^2k_{is}\sin(2\varphi_{i1})\\ &\quad + m_{i2}\xi_{i1}\Big(\xi_{i2}\Big(\sin(\varphi_{i1} - \varphi_{i2})\ddot\varphi_{i2} - \cos(\varphi_{i1} - \varphi_{i2})\dot\varphi_{i2}^2\Big)\Big)\\ &\quad - (m_{i1} + m_{i2})\xi_{i1}\cos(\varphi_{i1})\Big(A\omega^2\cos(\omega t) + g\Big) = 0,\\ &\quad i = 1\ldots 4. \end{aligned} \tag{10}$$

In our experiments, we use the rig with the set of two pairs of double pendulums shown in Figure 2. The vertical oscillations can be seen here as a blurry contour of the rig frame.

Before proceeding to numerical simulations and experiments concerning the motion of the pendula under vertical (parametric) excitation, some basic measurements of the

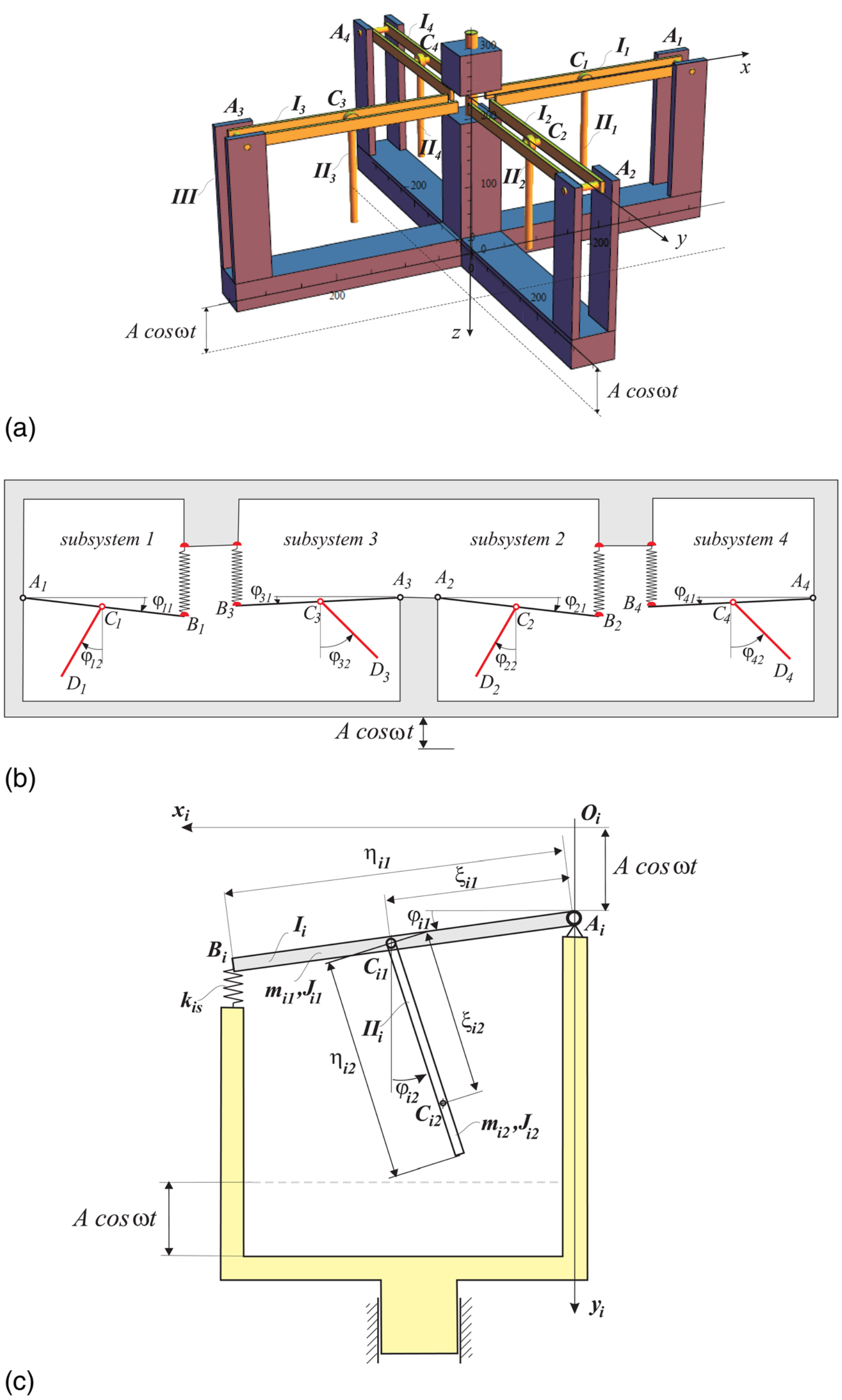

FIG. 1. Model of a set of ($n = 4$) pendula located at an oscillating platform (a) general view, ((b) and (c)) details describing system parameters.

individual pendulums have been carried out. The system parameters have been identified to be: $\xi_{i1} = 0.153$ [m], $\xi_{i2} = 0.096$ [m], $\eta_{i1} = 0.315$ [m], $\eta_{i2} = 0.145$ [m], $m_{i1} = 0.4$ [kg], $m_{i2} = 0.0166$ [kg]. The estimated values of the stiffness coefficients k_{is} are $k_{1s} = 4664$ [N/m], $k_{2s} = 4115$ [N/m], $k_{3s} = 4535$ [N/m], $k_{4s} = 4325$ [N/m]. (We measure the springs' deflections under several given weights to calculate an averaged value of k_{is}.) The values of the damping coefficients k_{ic} have been determined on the basis of the measured time interval in which the oscillations decay. The obtained values are as follows: $k_{1c} = 0.070$ [kg/ms], $k_{2c} = 0.035$ [kg/ms], $k_{3c} = 0.035$ [kg/ms], $k_{4c} = 0.050$ [kg/ms]. Our experimental system consists of four nonidentical subsystems (two pairs of double pendula). The pendula have the same lengths and masses but are suspended on springs with different stiffness and damping coefficients.

In designing the rig, we deliberately chosen nonidentical spring and damping elements (the differences in stiffness and damping coefficients are about 10% between the maximum and the minimum values). Our goal has been to check if the theoretically predicted synchronization of the nonidentical pendula[36,37] can be observed experimentally.

III. RESULTS AND DISCUSSION

In our experiment, the rig has been mounted on the shaker LDS V780 Low Force Shaker (basic data are as follows: sine force peak 5120 [N]; max random force (rms)

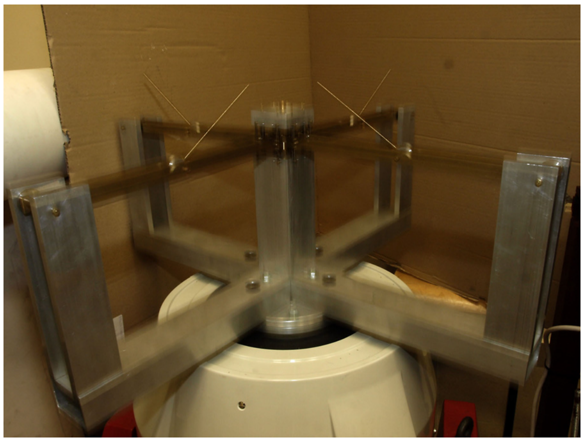

FIG. 2. Experimental rig.

4230 [N]; max acceleration sine peak $g_n = 111$; system velocity sine peak 1.9 [m/s]; displacement pk-pk $g_n = 25.4$ [mm]; moving element mass 4.7 [kg]). The shaker introduces practically kinematic periodic excitation $A\cos\omega t$, where A and ω are the amplitude and the frequency of the excitation, respectively. At initial moments the lower pendula have been assumed to be in the upper position, i.e., $\varphi_{2(1-4)} = \pi \pm \pi/36$. We fix the value of the excitation amplitude $A = 0.01 \pm 0.005$ [m] and consider excitation frequency ω as a control parameter.

For a qualitative classification of the pendula behavior, we use the following nomenclature; the pendula which rotate clockwise or counter-clockwise are marked by $+1$ and -1, respectively, the pendula which are at rest are marked by 0. The angular velocity of the pendulum is given as follows: $\varphi_{2i} = \omega t + b\sin(\omega t)$ for the case of clockwise rotation and $\varphi_{2i} = -\omega t + b\sin(-\omega t)$, where the harmonic component describes the influence of the gravity on the motion of pendula (b is constant for all pendula as their masses are the same).[36,37]

The most interesting case is when all four pendula rotate. In this case, one can observe various types of pendula synchronization. Typical examples of different types of synchronization are shown in Figures 3(a)–3(d), where yellow arrows indicate the direction of rotation. Figure 3(a) presents the case when all pendula rotate in the same direction, i.e., $(+1, +1, +1, +1)$. The pendula's displacements fulfill the relation $\varphi_{2i} - \varphi_{2j} = 0$, where $j = 1, 2, 3, 4$, $j \neq i$. In Figure 3(b), one observes the synchronous motion when 3 pendula (II_1, II_2, and II_3) rotate with the same direction, while the fourth in the opposite one—(+1,+1,+1,−1). In this case, $\varphi_{2(1-3)} + \varphi_{24} = 0$ and pendulum II_4 is in the state of mirror synchronization[37] with the cluster of synchronized pendula II_1, II_2, and II_3. In Figure 3(c), we present the variation of the case (+1,+1,+1,−1) when three pendula rotate in the same rotation velocity while the fourth one rotates twice slower. Pendula II_1, II_2, and II_3 are synchronized. The case when 2

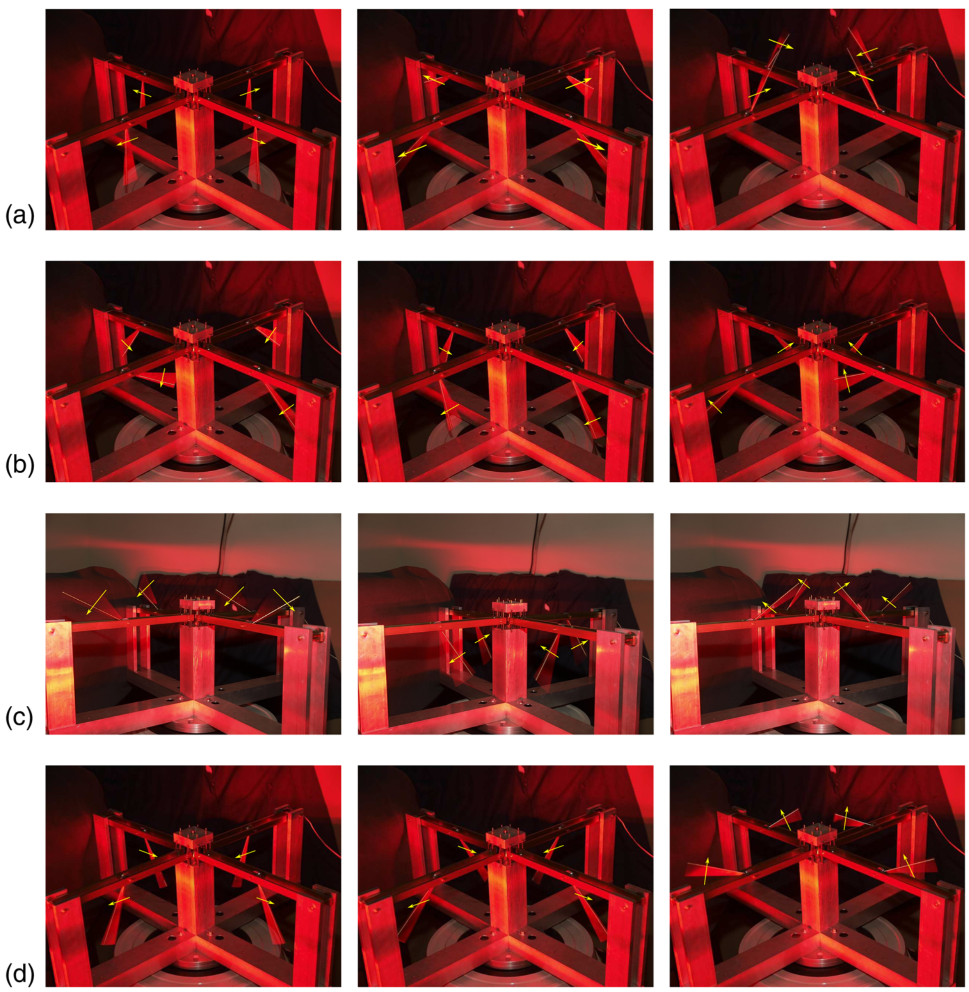

FIG. 3. Different types of experimentally observed synchronous states; (a) pendula rotate clockwise $(+1, +1, +1, +1)$, $\omega = 20.00$ [rad/s], (b) 3 pendula rotate clockwise while the fourth one counterclockwise $(+1, +1, +1, -1)$, $\omega = 24.00$ [rad/s], (c) 3 pendula rotate clockwise ($\omega = 29.00$ [rad/s]) while the fourth one counterclockwise $(+1, +1, +1, -1)$ with twice slower angular velocity, (d) 2 pendula rotate clockwise and 2 counterclockwise $(+1, +1, -1, -1)$, $\omega = 35.00$ [rad/s].

pendula rotate clockwise and two counterclockwise is presented in Figure 3(d). The pairs of the pendula which rotate in the same directions are synchronized and are in the state of cluster antiphase synchonization,[37] i.e., $\varphi_{2(1,2)} + \varphi_{2(3,4)} = \pi + 2b\sin(\omega t)$. In photo of Figures 3(a)–3(d), the camera has worked with shutter speed fixed and the flash fired at the second curtain, i.e., close to the moment when the shutter has been about to close. In such conditions, the observed trace of the pendula have created a blurry image with each pendulum highlighted with the flash. The highlighted image has appeared close to the end of exposure, and has allowed determination of the direction of the registered motion and rotating velocity ratio.

Experimental results can be compared with the numerical results obtained by the direct integration of Eqs. (4). Due to the lack of precise data from the experiments about the initial values of the angular velocity for the pendula (α), there exists a difficulty in direct comparison of both experimental a simulation data. In our calculations, we fixed the initial conditions $\varphi_{2i} = \pi, \dot{\varphi}_{2i} = 0, \varphi_{1i} = \pi/2, \dot{\varphi}_{1i} = 0$ and observe the type of behavior which a particular pendulum exhibits for given excitation parameter A and ω. Numerical results are shown in Figures 4(a)–4(e). In Figures 4(a)–4(d), red and navy blue colors indicate, respectively, clockwise and counterclockwise rotation, while the yellow one indicates that the pendulum is at rest. In the largest domain of the considered plane, pendula are in rest. Rotations is possible only in the small subsets indicated in blue and red so to keep pendula rotating both the amplitude A and frequency ω of the excitation should be chosen very accurately (practically to keep

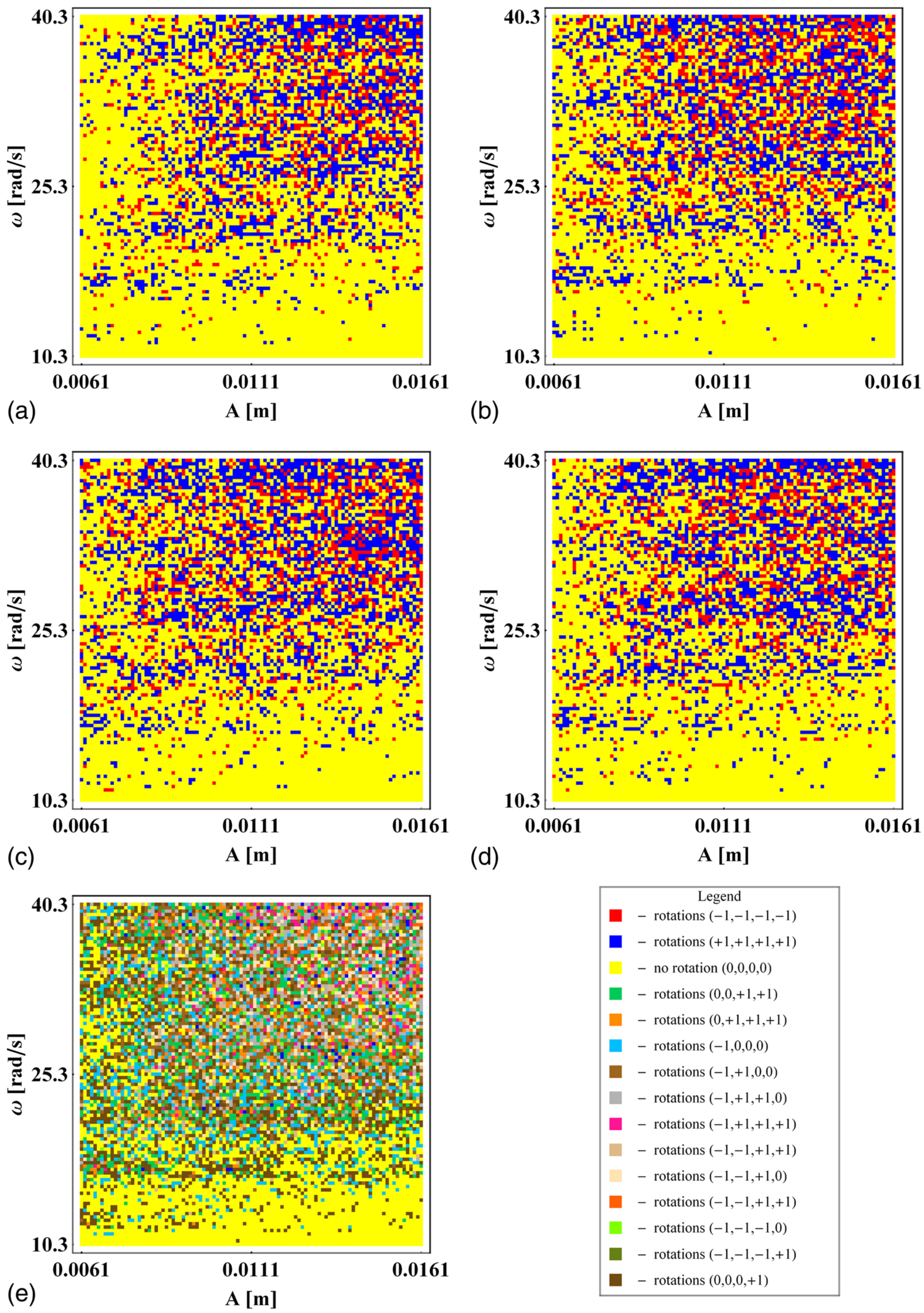

FIG. 4. Different types of pendula's behavior on the A-ω plane (a) pendulum II_1, (b) pendulum II_2, (c) pendulum II_3, (d) pendulum II_4, red—pendulum rotates clockwise, navy blue—pendulum rotates counterclockwise, yellow—pendulum at rest, (e) combined synchronous states of 4 pendula.

pendula rotating one has to adopt appropriate controlling procedure). Figure 4(e) summarizes the results obtained for the particular pendula and shows how all pendula behave. Notice that the type of pendula synchronization very strongly depends on the excitation parameters. Very small changes of these parameters (smaller than the accuracy of our measurement) can lead to the change of synchronization configuration. The structure of the basins of different synchronous configurations is similar to that obtained for the result of the dice throw[41,42] and explains the difficulty in the predictability of the synchronous structure.

IV. CONCLUSIONS

In the considered system consisting of a set of four double pendula mounted on the platform which can oscillate vertically, one can observe the synchronous states of both clockwise and counter-clockwise rotating pendula. In the experiment using simple mechanical rig, we confirmed the existence of different types of synchronous configurations of rotating nonidentical pendula and the extreme sensitivity of the synchronized state on the system parameters and the initial conditions. This sensibility introduces pseudo-randomness to the predictability of the synchronous state. Rotating pendula can be 1:1 and 1:2 synchronized with the oscillations of the platform. The existence of experimentally observed synchronous states is confirmed in the numerical simulations. Generally, synchronous rotation of pendula is robust as it exists for the wide range of excitation parameters, but particular synchronous states are very sensitive to the changes of system parameters. In practical application to the extraction energy from the sea waves, one has to apply a feedback control mechanism. This mechanism should be capable of sensing the roughness of the waves in the sea and keep the pendula rotating permanently in the desired synchronous configuration, therefore, providing energy for uninterrupted power generation.

ACKNOWLEDGMENTS

This work has been supported by the Polish Department for Scientific Research (currently National Center for Science) under Project No. N N501 249238.

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