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Review

Dynamics of axially moving continua

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ABSTRACT

Dynamics of axially moving material continuum is investigated for over 60 years. Research interest in this subject is still high due to more and more new fields of application. Alongside traditional applications, such as band saws' blades and axially moving paper webs, appeared as flat objects moving at high speeds in space. Recently many studies on the dynamics of axially moving beam-like and plate-like systems were published. Inclusion in analysis viscoelastic properties of broad moving continua is also the subject of research. The paper presents a review of research in this field with particular emphasis on the axially moving plate-like elastic and viscoelastic systems. Unlike the string-like and beam-like systems, such review on the plate-like systems has not been published yet. A brief overview of the most important studies on the dynamics of moving string-like and beam-like systems is presented as well. This paper also includes a comparative analysis of some results of studies published by the authors in the field of dynamics of axially moving viscoelastic systems. Some suggestions on the directions of further research in this field end the paper.

Contents

1. Introduction

A material continuum moving axially at high speed can be met in numerous different technical applications. These comprise band saws, web papers during manufacturing, processing and printing processes, textile bands during manufacturing and processing, pipes transporting fluids, transmission belts as well as flat objects moving at high speeds in space. In all these so varied technical applications, the maximum transport speed or the transportation speed is aimed at in order to increase efficiency and optimize investment and performance costs of sometimes very expensive and complex machines and installations. The dynamic behavior of axially moving systems very often hinders from reaching these aims.

Dynamics of axially moving systems has been investigated for over 60 years already. In the early 1950s, first publications including results of investigations of transverse vibrations of an axially moving string are found. Those papers were authored by Sack [63] and Archibald and Emslie [1]. In the early stage, mainly

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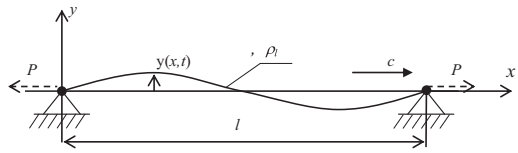

Fig. 1. Model of axially moving string.

string-like and beam-like axially moving systems were the objects of interest for scientists. Since then, a vivid interest of researchers in axially moving system can be observed. This interest supported by newer and newer fields of applications (e g. in outer-space) throughout years remains firm on a high level till today.

The dynamic behavior of axially moving plate-like systems has been the least recognized during long time. Since in the last time many studies in dynamics of moving plate-like systems were published, the authors decided to present and reorder them in this review. Unlike the string-like and to a lesser extent beam-like systems, such review on the axially moving plate-like systems has not been published yet.

This paper is divided into seven sections. In Section 2, a brief overview of the most important papers on the dynamics of moving string-like systems is presented. In Section 3 similar overview on beam-like systems is presented. The state of knowledge on dynamic investigations of axially moving elastic plate systems is presented in Section 4. Section 5 shows significant studies of the axially moving viscoelastic plate systems Section 6 includes a comparative analysis of some results of studies on dynamics of moving viscoelastic systems published by the authors. Section 7 shows directions of future studies in the field of dynamics of the axially moving materials.

2. Short review of studies of string-like systems

Among earliest studies devoted to the linear analysis of vibrations of a moving string, the most frequently quoted is the study by Archibald and Emslie [1]. It presents the results of investigations on transverse vibrations of a string moving with a constant speed c along the longitudinal direction, denoted by x (Fig. 1).

Starting from the Hamilton's principle, the already classical differential equation with partial derivatives has been expressed as

$$\frac{\partial^2 y}{\partial t^2}+2\,c\,\frac{\partial^2 y}{\partial t\,\partial x}-\left(\frac{T-\rho_l\,c^2}{\rho_l}\right)\frac{\partial^2 y}{\partial x^2}=0 \tag{1}$$

where ρ_l is the mass of the string unit length, T is the longitudinal tension force.

A solution to Eq. (1) has been presented in the form of a sum of harmonic functions as follows:

$$y=C_1\,\cos\,\omega\left[t+\left(\frac{c\rho_l+\sqrt{\rho_l T}}{T-\rho_l c^2}\right)x\right]+C_2\,\cos\,\omega\left[t+\left(\frac{c\rho_l-\sqrt{\rho_l T}}{T-\rho_l c^2}\right)x\right] \tag{2}$$

where C_1, C_2 are constants, ω is the angular frequency of transverse vibrations.

Assuming the simple boundary conditions $y|_{x=0}=0,\ y|_{x=l}=0$ on the basis of solution (2), an analytical dependence of the transverse vibration frequency of the string f_n on its transport speed c has been determined:

$$f_n=\frac{\omega}{2\pi}=\frac{n(T-\rho_l c^2)}{2l\sqrt{\rho_l T}};\quad n=1,\ 2,\ 3.... \tag{3}$$

In the same study [1], the authors have made one more important discovery. By introducing the string wave propagation speed $c_w=(T/\rho_l)^{1/2}$ into the string motion Eq. (1), they have shown that a solution to the motion equation can be expressed in the general form as

$$y=F_1[x-(c_w+c)\,t]+F_2[x+(c_w-c)\,t] \tag{4}$$

where F_1 and F_2 are functions describing propagation of waves in opposite directions along the axis x.

The presented study [1], as well as later works showing the results of investigations of transverse vibrations of axially moving string systems within the linear theory, allows us to draw the following conclusions:

(a) If c_w is the speed of wave propagation measured by an observer moving together with the string, and c is the speed of displacement of the string, then disturbances propagate with the speed c_w+c along the direction the string moves and with the speed c_w-c along the direction opposite to the string motion (both the speeds are determined with respect to the still observer).
(b) A difference in the speeds causes that the motion of the string is characterized by a variable angle of the phase shift; the distortions that displace along the direction opposite to the string motion are in a phase lag with respect to the distortions that propagate along the string motion direction.
(c) Frequency of natural vibrations decreases with an increase in the string transport speed; when $c=c_w$, a divergent-type instability of motion occurs.

An influence of damping on the dynamic behavior of the moving string within the linear theory has been investigated in several studies. In [53] there is a linear damping force in the form of $\beta\ w_{,t}$, where β denotes a viscous damping coefficient and the subscript denote the partial differentiation. In [38,69] damping in the form of $\beta\ (w_{,t}+cw_{,x})$ has been considered. The quantity $(w_{,t}+cw_{,x})$ represents a transverse speed of the string element, measured by a still observer, $w_{,t}$ is an analog speed measured by an observer moving together with the string. In all these works, the authors have investigated changes in complex natural frequencies and amplitudes of system vibrations in relation to the string transport speed. The system motion stability has been found to be dependent on the way the dissipative force is modeled and on the value of the damping coefficient.

In several studies the elastic string models moving with a variable axial speed have been analyzed (e.g. [2,57]). Positive acceleration of the transport motion of the string model has a stabilizing effect on its transverse vibrations. Negative acceleration of the string transport motion acts just the opposite. Then, the

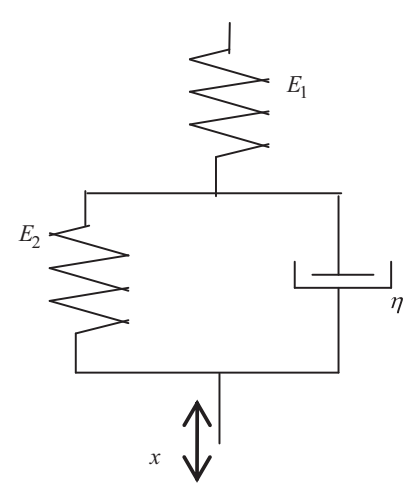

Fig. 2. Poynting–Thompson rheological model.

occurring destabilization of transverse vibrations can be compared to the effect of negative damping.

For the first time the dynamic response of a viscoelastic moving string was discussed in the paper by Fung et al. [17]. The string material was assumed to be constituted by the hereditary integral type. The governing equations in the form of nonlinear differential–integral equations were solved by the finite difference method. Shortly afterwards, the nonlinear problem of transverse vibrations of a viscoelastic string moving with time-dependent speed in a state of uniform initial stress was studied by the same authors [18].

It is worth to take a look at the derivation of the mathematical model in [18], because it is a typical approach seen in many later studies on various viscoelastic objects. Physical model of the system considered in [18] is shown in Fig. 1. Firstly the equation of motion in the y direction was presented in general form:

$$\left(\frac{P}{A}+\sigma\right)\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial \sigma}{\partial x}=\rho\frac{\partial^2 v}{\partial t^2}, \tag{5}$$

where σ is the perturbed axial stress, v is the displacement in the transverse direction, P is the uniform initial tension force, A is the area of cross section of the string, ρ is the density of the spring material.

The Lagrangian strain component in the x direction (Fig. 1) is related to the displacement v in the following way:

$$\varepsilon(x,t)=\frac{1}{2}\left(\frac{\partial v(x,t)^2}{\partial x}\right). \tag{6}$$

In the derivation of the equations of motion the Poynting–Thompson rheological model of the string material was used (Fig. 2).

The one-dimensional constitutive equation of a differential type material obeys the following relation:

$$\sum_{j=0}^{R} a_j\frac{d^j\sigma}{dt^j}=\sum_{j=0}^{S} b_j\frac{d^j\varepsilon}{dt^j},\quad (a_0\neq 0,\ \ b_0=1) \tag{7}$$

Eq. (7) may be written in the form:

$$\mathbf{A}\,\sigma=\mathbf{B}\,\varepsilon \tag{8}$$

where **A** and **B** are differential operators defined as

$$\mathbf{A}=\sum_{j=0}^{R} a_j\frac{d^j}{dt^j};\quad \mathbf{B}=\sum_{j=0}^{S} b_j\frac{d^j}{dt^j} \tag{9}$$

In the case of the three-parameter Poynting–Thompson rheological model the differential constitutive law of viscoelastic material can be written as

$$\frac{\partial\sigma}{\partial t}+\frac{E_1+E_2}{\eta}\sigma=E_1\frac{\partial\varepsilon}{\partial t}+\frac{E_1E_2}{\eta}\varepsilon. \tag{10}$$

Comparing Eqs. (10) and (7) the following relations are obtained:

$$a_0=\frac{E_1+E_2}{\eta};\quad a_1=1;\quad b_0=\frac{E_1E_2}{\eta};\quad R=S=1. \tag{11}$$

Multiplying Eq. (5) with operator **A**, and using Eqs. (6), (8) and (11), the nonlinear mathematical model of viscoelastic string with varying transport speed has the following form:

$$\begin{aligned}&\frac{P}{A}[a_0v_{xx}+a_1(v_{xxt}+x_tv_{xxx})]+\frac{3}{2}b_0v_x^2v_{xx}\\&\quad+b_1(2v_xv_{xx}v_{xt}+2v_xv_{xx}^2x_t+v_x^2x_{xxt}+v_x^2v_{xxx}x_t)\\&=\rho a_0(v_{tt}+2x_tv_{xt}+x_t^2v_{xx}+x_{tt}v_x)\\&\quad+\rho a_1(v_{ttt}+3x_tv_{ttx}+3x_t^2v_{xxt}+x_t^3v_{xxx}+3x_{tt}v_{xt}+3x_tx_{tt}v_{xx}+x_{ttt}v_x)\end{aligned} \tag{12}$$

In Eq. (12) the subscripts denote the partial differentiation. Eq. (12) was derived under the assumptions that the perturbed stress $\sigma(x,t)\ll P/A$ and the perturbed stress gradient $\sigma_x(x,t)$ is small and can be neglected.

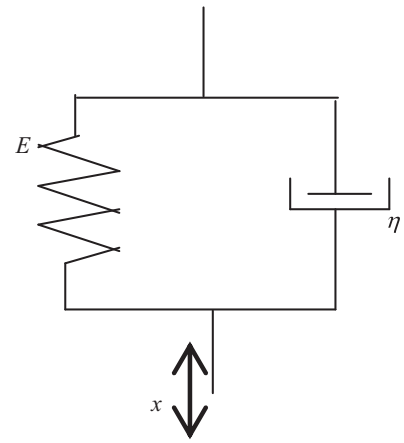

Fig. 3. Kelvin–Voigt rheological model.

Since the publication of the works by Fung et al., dynamic analysis of viscoelastic strings with varying transport speeds in a state of uniform stress, continuously interested various researchers (e.g. [9,19]). However, such assumptions do not reflect the dynamic behavior of actual objects. Variable tension of the string, as the cause of parametric vibrations, was early noticed by various researchers. Particularly worth paying attention to the publication by Mockersturm and Guo [48], where it was carried out a critical discussion of previous results published in the works [12,13,80,81]. In the study [48] authors note that while the previous works provided a method to include material damping in the analysis, an important material dissipation mechanism was not included in the derivation of the equation of motion.

The weekly nonlinear equation of motion for an axially moving string composed of a viscoelastic material is derived in [48]. The string is moving with a constant speed v under parametric excitation. The source of parametric excitation in a tensioned string is a moving support which causes the tension to change but has negligible effect on the total length between the supports. Assuming the Kelvin–Voigt rheological model of the string material (Fig. 3), the tension force is related to the strain ε by

$$T=A\,(E\,\varepsilon+\eta\,\dot{\varepsilon})=A\left[E\,\varepsilon+\eta\left(\frac{\partial\varepsilon}{\partial t}+v\frac{\partial\varepsilon}{\partial x}\right)\right], \tag{13}$$

where A is the cross-section area of the string, E is the elastic modulus, η is the viscous material constant.

The strain in the string is varying harmonically $\varepsilon=\Delta\sin(\omega t)$, then the dynamic tension in the string is given by

$$\frac{T}{A}=E\Delta\ \sin\ (\Omega t)+\eta\Delta\Omega\ \cos\ (\Omega t)=\Lambda\ \sin(\Omega t+\phi), \tag{14}$$

where $\Lambda^2=(E\Delta)^2+(\eta\Delta\Omega)^2$. Both the excitation amplitude Δ and the frequency Ω affect the amplitude of parametric excitation.

The axial and transverse displacements of the spring are denoted by U and W, respectively. The Lagrangian strain in the string is

$$\varepsilon(x,t)=\frac{\partial U}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 \tag{15}$$

Substituting Eq. (15) into Eq. (13), the tension in the viscoelastic material is

$$\frac{T}{A}=E\left[\frac{\partial U}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right]+\eta\left(\frac{\partial^2 U}{\partial x\,\partial t}+\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x\,\partial t}+v\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2}\right) \tag{16}$$

The axial strain $\partial U/\partial x$ is assumed to be composed of a constant strain and a harmonically varying strain

$$\frac{\partial U}{\partial x}=\varepsilon_0+\varepsilon_1\ \cos(\Omega t) \tag{17}$$

Taking into account nondimensional parameters

$$w=\frac{W}{L};\quad \xi=\frac{X}{L};\quad c=v\sqrt{\frac{\rho}{E\varepsilon_0}};\quad \tau=t\sqrt{\frac{E\varepsilon_0}{\rho L^2}};\quad a=\frac{\varepsilon_1}{\varepsilon_0};$$
$$\omega=\Omega\sqrt{\frac{\rho L^2}{E\varepsilon_0}};\quad \varsigma=\eta\sqrt{\frac{1}{\rho E\varepsilon_0 L^2}};\quad \varphi=\frac{EA}{T_0}, \tag{18}$$

the dimensionless equation of the transverse motion of the viscoelastic string is

$$\begin{aligned}\frac{\partial^2 w}{\partial\tau^2}+2c\frac{\partial^2 w}{\partial\xi\partial\tau}+\left[c^2-1-a\ \cos(\omega\tau)+\varsigma\omega a\ \sin(\omega\tau)\right]\frac{\partial^2 w}{\partial\xi^2}\\ =\frac{3}{2}\varphi\left(\frac{\partial w}{\partial\xi}\right)^2\frac{\partial^2 w}{\partial\xi^2}+\varsigma\frac{\partial w}{\partial\xi}\left(2\frac{\partial^2 w}{\partial\xi^2}\frac{\partial^2 w}{\partial\xi\partial\tau}+\frac{\partial w}{\partial\xi}\frac{\partial^3 w}{\partial\xi^2\partial\tau}\right)\\ +c\varsigma\frac{\partial w}{\partial\xi}\left[2\left(\frac{\partial^2 w}{\partial\xi^2}\right)^2+\frac{\partial w}{\partial\xi}\frac{\partial^3 w}{\partial\xi^3}\right]\end{aligned} \tag{19}$$

Authors of the study [48] note that three components of the equation of motion (19) are not included in the analogous equation in [12,13,80,81], where the same dynamic objects are investigated. This applies to the last component of the left side of Eq. (19) and the two last components of the right side of the same equation. The results presented in [48] show how the omission of these components resulted in nonlinear dynamic behavior of the studied system. Because the goal of the study [48] is to investigate the importance of the mechanics neglected in previous works, the same solution procedure of the direct perturbation method of multiple scales is applied.

Studies carried out in [48] allowed us to state the way that viscoelastic material behavior was previously incorporated in the mathematical model is only correct if the string is not translating. When the string is moving, the material time derivative is not simply the partial time derivative, a steady-state component, proportional to the transport speed, and the axial strain gradient, also appears. This steady-state dissipation significantly effects the frequencies at which nontrivial limit cycles exist and also amplitudes of these limit cycles. When the transport speed of the string increases, the discrepancy between the compared results grows. Figs. 4 and 5 illustrate these conclusions, showing examples of results presented in the study [48].

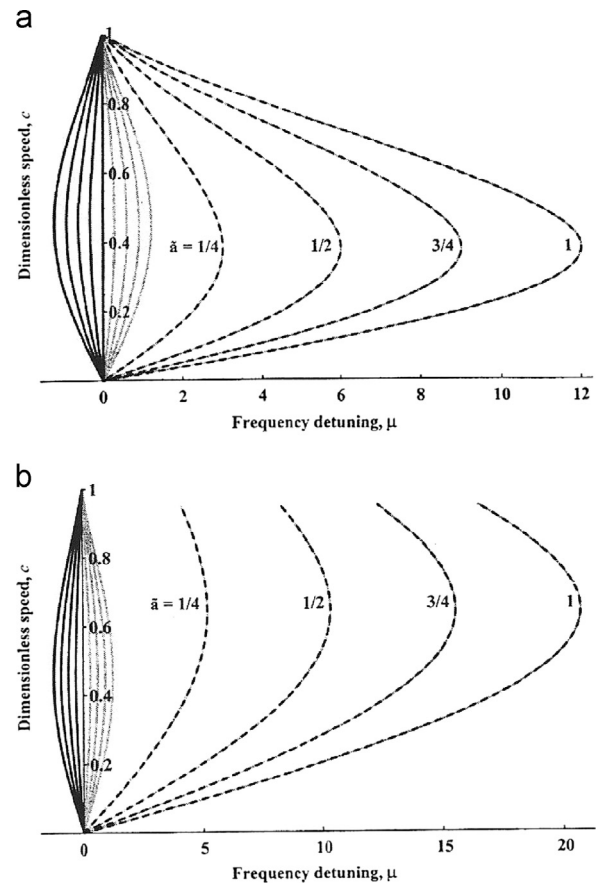

Fig. 4. Stability boundaries of the first combination parametric resonance obtained with (a) and without (b) steady-state dissipation [48].

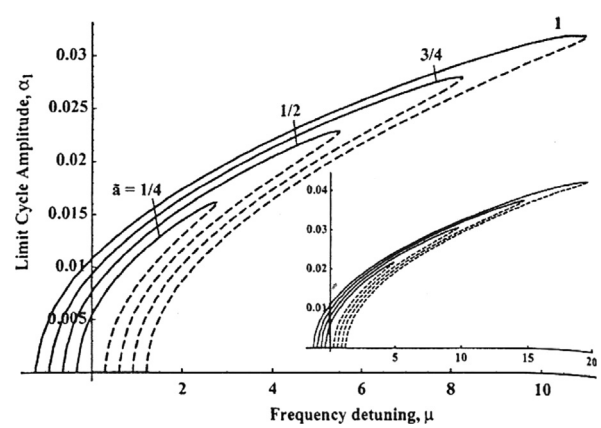

Fig. 5. The amplitude of the nontrivial limit cycles in the first combination parametric resonance. Inset shows solution neglecting steady state dissipation [48].

3. Short review of studies of beam-like systems

In the initial period, only operation of band saws inspired researchers in their investigations of the beam model [50]. While cutting wood or metal with band saws, the saw blade is subjected to edge loading forces. These forces generate both a transverse motion and a torsional motion of the blade and can lead to the band buckling. In early studies, torsional vibrations of the axially moving beam loaded with a force focused on the edge were investigated, determining thus a dependence between the torsional vibration frequency and the load. However, later in [52] a significant error was found that is made while neglecting a coupling between torsional and transverse vibrations of the beam. An analysis of the uncoupled system leads to overrated values of critical buckling load, which finally yields wrong results in analysis of the system motion stability.

In first works devoted to axially moving beam systems, transverse vibrations of the Bernoulli beam rigidly supported were analyzed. Similarly as in the case of the moving string, the transverse motion of the beam is characterized by a variable angle of the phase displacement. Due to dissipation of energy in the beam, divergent-type instability occurs for individual modes of vibrations at various critical transport speeds. Beam support flexibility and the normal component of acceleration of the beam system in its motion around guide rolls cause the stresses in the axially moving beam to be a function of its transport speed [50]. The Timoshenko equations for the axially moving beam were generated for the first time in the study by Simpson [65]. A comparison of the results obtained for both the models, i.e. the Bernoulli beam and the Timoshenko beam shows their good convergence only for low transport speeds. The value of critical

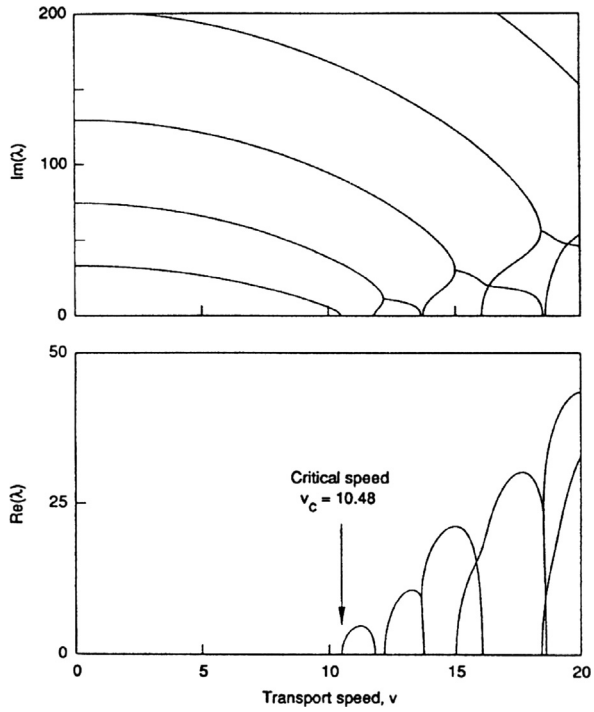

Fig. 6. Frequency spectrum of a simply supported traveling beam [73].

speed determined with the Timoshenko beam model is lower than with the Bernoulli beam model.

In the early 1990s, Wickert and Mote [73] published their study, in which the conclusions resulting directly from the previously issued works on string systems and beam systems within the linear theory were employed. Treating an object as a gyroscopic system and not taking any simplified assumptions, the authors investigated free and forced vibrations with the modal analysis introduced by Meirovith [47] for discrete gyroscopic systems and by D'Eleuterio and Huges [14] for continuous gyroscopic systems. To solve the linear problem, Green functions were used. The authors showed that each eigenfrequency of the system decreased with an increase in the transport speed. Eigenfrequencies, despite their conservative nature, have a complex form due to the fact that there is a transport speed. A very characteristic diagram that illustrates the dynamic behavior of linear axially moving systems in the form of a dependence of eigenfrequencies of transverse vibrations on the transport speed, which can be found in the study [73], is presented in Fig. 6. A solution to the linear problem of the axially moving string and beam presented in [73] has become a traditional approach, to which many references on the topic issued after its publications, refer to.

Works devoted to investigations of nonlinear vibrations of moving beams constitute a very important chapter in the studies on the dynamic behavior of axially moving systems. Nonlinear investigations of the moving beam under tension are motivated by two clearly visible drawbacks of the linear theory. Each of them can be noticed when the transport speed increases. The first drawback can be seen at velocities lower than the first (undercritical) wave speed of the beam. Much the same as in the case of the moving string, a participation of nonlinearity in beam displacements increases with an increase in the transport speed, thus imposing some limitations of an application of the linear analysis with respect to a low amplitude and a low transport speed. The earliest calculations of the fundamental period of nonlinear transverse vibrations were made by Mote [51] for a limited case of vanishing flexural stiffness. Calculation problems in integration of the motion equation restricted the solution to the speed below

Fig. 7. Model of axially moving beam considered in [72].

40% of the critical speed, but the results were sufficient to show that an effect of changes in the tension force during vibrations increased significantly with displacement.

The second drawback of the linear theory can be seen in the range of overcritical transport speeds of the moving beam. Then, the nontrivial equilibrium position bifurcates from a simple configuration and either a local motion around the equilibrium position or a global motion between the co-existing equilibrium positions can occur. In the system of the initially straight pipe with a moving fluid, it has been demonstrated in studies of vibration of the pipeline, which contains moving liquid. Holmes [24] investigated precisely the symmetrical saddle-node bifurcation, which occurs at the critical speed of the liquid. For transport speeds higher than the critical speed, a stable limit cycle does not occur when nonlinearity is taken into consideration, and solutions disappear in time up to the stable half-sinusoidal equilibrium position. A considerably different dynamic behavior in the overcritical range occurs in the case of a cantilever pipe with a moving fluid. At the critical speed, the trivial solution bifurcates into a periodic orbit and is subject to a Hopf bifurcation. These results and next of the conducted dynamic analyses have allowed early enough for drawing a conclusion that it is impossible to obtain an analytical description of nonlinear vibrations of axially moving systems that can be employed within the whole range of transport speeds. In the literature, the motion limitations were defined and simplified assumptions that tended to cover only most significant features of the system began to appear in the mathematical description of the systems under analysis. These tendencies are well illustrated in the publication by Wickert [72]. The author deals with transverse and longitudinal displacements of the beam moving at a constant speed. A schematic view of the system under consideration is depicted in Fig. 7.

Two coupled, nonlinear differential equations of the beam motion along the longitudinal and transverse directions were formulated with the use of Hamilton's principle

$$\begin{aligned}&(u'_{tt}+2vu'_{xt}+v^2u'_{xx})-v_l^2\left(u'_x+\tfrac{1}{2}w'^2_x\right)'_x=0\\&(w'_{tt}+2vw'_{xt}+v^2w'_{xx})-\left\{\left[1+v_l^2\left(u'_x+\tfrac{1}{2}w'^2_x\right)\right]w'_x\right\}'_x+v_f^2w'_{xxxx}=0\end{aligned}\tag{20}$$

where $v=c\sqrt{\rho A/P}$; $v_l=\sqrt{EA/P}$; $v_f=\sqrt{EI/Pl^2}$.

Next, the author introduced a simplification that consists in expressing the dynamic component of the longitudinal stress in second equation (20) as a function of time only into the system of equations. He refers here to the results of experiments on the basis of which it was stated that longitudinal perturbations propagated much faster than the transverse ones within the technologically usable range of beam model parameters [50]. A steel band saw with a blade of the length of $l=2$ m and a cross-section of 300 mm × 2 mm was analyzed. This blade subject to tensioning with a force of $P=26$ kN is characterized by the longitudinal phase speed of 5100 m/s, whereas the speed of transverse vibrations corresponding to the lowest mode is equal to 75 m/s. The typical working speed of such a saw equals 50 m/s. On the time diagrams of low transverse modes of vibrations, changes in stresses are almost instantaneous as the effect of longitudinal inertia is low. Taking this into account and having neglected time derivatives and

the term $v^2\, u_{,xx}$ as well, the author integrated the first equation of motion (20). The obtained solution points out to the fact that the longitudinal displacement increases within a limited range for finite transverse vibrations. Thus, a simplified form of the differential–integral equation of the beam transverse motion has been defined in the following form:

$$(w_{,tt}+2vw_{,xt}+v^2w_{,xx})-w_{,xx}+v_w^2w_{,xxxx}=\frac{v_l^2}{2}w_{,xx}\int_0^1 w_{,x}^2\,dx \tag{21}$$

It should be remarked that the governing Eqs. (21) and (5) are based on different assumptions on the longitudinal motion. This point has been emphasized in the publication by Chen and Ding [7]. To obtain an asymptotic solution to the differential–integral equation of motion (21), in the study [72] a modal theory of perturbations with the Krylov–Bogolubov method has been employed. The assigned points of motion Eq. (21) have been examined analytically on the assumption of the following bifurcation configuration:

$$w^{(k)}=\pm\frac{2}{k\pi v_1}\sqrt{v^2-v^{(k)2}}\ \sin\ (k\pi x);\quad k=1,\ 2,\ \ldots \tag{22}$$

Each configuration $w^{(k)}$ exists only if v exceeds the kth critical speed defined by the following relationship:

$$v^{(k)}=\sqrt{1+(k\pi v_w)^2} \tag{23}$$

In this solution Wickert has shown that for the range of undercritical velocities, the linear theory underestimates an occurrence of the instability region, whereas for overcritical transport velocities, there is an overestimation. Fig. 8 illustrates this regularity. The results presented in the study by Wickert [72] point out to the fact that it is the most essential to include the geometrical nonlinearity in the investigations of the moving material continuum in the neighborhood of critical speeds. In these regions, low modal stiffness is dominated by nonlinear longitudinal stiffness.

At the turn of the last and present century, the studies by Pakdemirli and collaborators were published, where the dynamics of axially moving beam with variable speed was investigated [55,56,58]. In these works the direct perturbation method of multiple scales to solve the nonlinear model was used for the first time. Since then, the method of multiple scales is the most frequently used method in the dynamic analysis of nonlinear moving systems. Based on the work [55], this method is presented in detail.

In [55] the dimensionless equation of motion of the beam derived in the study by Wickert [72] is considered:

$$\frac{\partial^2 w}{\partial t^2}+2\,v\frac{\partial^2 w}{\partial x\,\partial t}+\frac{\partial w}{\partial x}\frac{\partial v}{\partial t}+v_f^2\frac{\partial^4 w}{\partial x^4}+(v^2-1)\frac{\partial^2 w}{\partial x^2}=0, \tag{24}$$

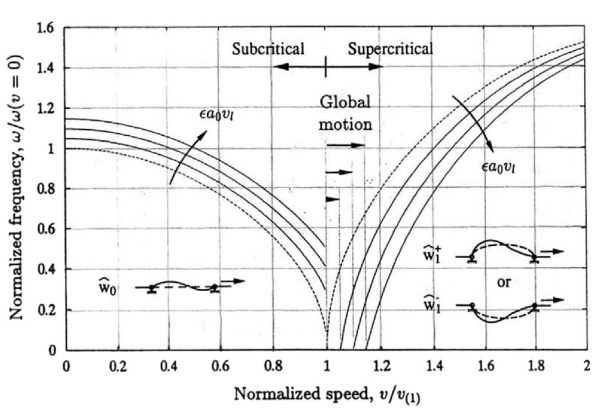

Fig. 8. Linear and nonlinear fundamental frequencies of the moving beam [72].

with the simple boundary conditions

$$w(0,t)=w(1,t)=0;\quad \frac{\partial^2 w(0,t)}{\partial x^2}=\frac{\partial^2 w(1,t)}{\partial x^2}=0, \tag{25}$$

where w is the transverse displacement of the beam; v is the axial velocity, v_f is the dimensionless flexural stiffness (order one component).

Transport speed is changing harmonically around a constant value v_0

$$v=v_0+\varepsilon v_1\ \sin(\Omega t), \tag{26}$$

where ε is the small parameter, εv_1 is the small parameter representing the amplitude of speed change, Ω is the frequency of speed change.

Substituting Eq. (26) into Eq. (24) and keeping up the components of the first order of approximation gives

$$\frac{\partial^2 w}{\partial t^2}+2v_0\frac{\partial^2 w}{\partial x\,\partial t}+(v_0^2-1)\frac{\partial^2 w}{\partial x^2}+v_f^2\frac{\partial^4 w}{\partial x^4}+\varepsilon\left[v_1\Omega\ \cos(\Omega t)\frac{\partial w}{\partial x}+2v_1\ \sin(\Omega t)\frac{\partial^2 w}{\partial x\partial t}+2v_0v_1\ \sin(\Omega t)\frac{\partial^2 w}{\partial x^2}\right]=0 \tag{27}$$

To solve Eq. (27) the direct-perturbation method of multiple scales was applied in the publication by Öz and Pakdemirli [55]. Then the first-order solution has the following form:

$$w(x,\ t;\ \varepsilon)=w_0(x,\ T_0,\ T_1)+\varepsilon w_1(x,\ T_0,\ T_1), \tag{28}$$

where w_0 is the displacement function of order 1, w_1 is the displacement function of order ε; $T_0=t$ is the fast time scale, $T_1=\varepsilon t$ is the slow time scale,

In terms of the new variables, the time derivatives can be presented as

$$\frac{d}{dt}=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}+\cdots;\quad \frac{d^2}{dt^2}=\frac{\partial^2}{\partial T_0^2}+2\varepsilon\frac{\partial}{\partial T_0}\frac{\partial}{\partial T_1}+\cdots. \tag{29}$$

Substituting (28) and (29) into Eq. (27) and separating the components of each order of ε gives

$$Order\ \ (1):\quad \frac{\partial^2 w_0}{\partial T_0^2}+2v_0\frac{\partial^2 w_0}{\partial T_0\partial x}+(v_0^2-1)\frac{\partial^2 w_0}{\partial x^2}+v_f^2\frac{\partial^4 w_0}{\partial x^4}=0 \tag{30}$$

$$\begin{aligned}Order\ \ (\varepsilon):\quad &\frac{\partial^2 w_1}{\partial T_0^2}+2v_0\frac{\partial^2 w_1}{\partial T_0\partial x}+(v_0^2-1)\frac{\partial^2 w_1}{\partial x^2}+v_f^2\frac{\partial^4 w_1}{\partial x^4}\\ &=-2\frac{\partial w_0}{\partial T_0}\frac{\partial w_0}{\partial T_1}-2v_0\frac{\partial^2 w_0}{\partial T_1\partial x}-2v_1\ \sin(\Omega T_0)\frac{\partial^2 w_0}{\partial T_0\partial x}\\ &\quad-2v_0v_1\ \sin(\Omega T_0)\frac{\partial^2 w_0}{\partial x^2}-\Omega v_1\ \cos(\Omega T_0)\frac{\partial w_0}{\partial x}\end{aligned} \tag{31}$$

The first-order solution of Eq. (30) can be written as follows:

$$w_0(x,\ T_0,\ T_1;\varepsilon)=A_n(\ T_1)\ e^{i\omega_n T_0}Y_n(x)+\overline{A}_n(\ T_1)\ e^{-i\omega_n T_0}\overline{Y}_n(x) \tag{32}$$

The spatial functions $Y_n(x)$ satisfy the characteristic equation:

$$v_f^2\frac{\partial^4 Y_n}{\partial x^4}+(v_0^2-1)\frac{\partial^2 Y_n}{\partial x^2}+2iv_0\omega_n\frac{\partial^2 Y_n}{\partial x}-\omega_n^2Y_n=0 \tag{33}$$

and the boundary conditions

$$Y_n(0)=0;\quad Y_n(1)=0;\quad \frac{\partial^2 Y_n(0)}{\partial x^2}=0;\quad \frac{\partial^2 Y_n(1)}{\partial x^2}=0 \tag{34}$$

The solution of Eq. (33) is

$$Y_n(x)=c_{1n}\left(e^{i\beta_{1n}x}+C_{2n}e^{i\beta_{2n}x}+C_{3n}e^{i\beta_{3n}x}+C_{4n}e^{i\beta_{4n}x}\right) \tag{35}$$

The coefficients β_{in} satisfy the dispersive equation:

$$v_f^2\beta_{in}^4+(v_0^2-1)\beta_{in}^2-2v_0\omega_n\beta_{in}-\omega_n^2=0,\quad i=1,2,3,4;\quad n=1,\ 2\ldots \tag{36}$$

Applying the boundary conditions (34) to the solution, the matrix equation is obtained:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta_{1n}^2 & \beta_{2n}^2 & \beta_{3n}^2 & \beta_{4n}^2 \\ e^{i\beta_{1n}} & e^{i\beta_{2n}} & e^{i\beta_{3n}} & e^{i\beta_{4n}} \\ \beta_{1n}^2 e^{i\beta_{1n}} & \beta_{2n}^2 e^{i\beta_{2n}} & \beta_{3n}^2 e^{i\beta_{3n}} & \beta_{4n}^2 e^{i\beta_{4n}} \end{bmatrix} \begin{bmatrix} 1 \\ C_{2n} \\ C_{3n} \\ C_{4n} \end{bmatrix} c_{1n} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{37}$$

For nontrivial solutions, the characteristic determinant of Eq. (37) must be zero, which yields the support condition in the following form:

$$\begin{aligned} &\left(e^{i(\beta_{1n}+\beta_{2n})}+e^{i(\beta_{3n}+\beta_{4n})}\right)\left(\beta_{1n}^2-\beta_{2n}^2\right)\left(\beta_{3n}^2-\beta_{4n}^2\right) \\ &+\left(e^{i(\beta_{1n}+\beta_{2n})}+e^{i(\beta_{2n}+\beta_{4n})}\right)\left(\beta_{2n}^2-\beta_{4n}^2\right)\left(\beta_{3n}^2-\beta_{1n}^2\right) \\ &+\left(e^{i(\beta_{2n}+\beta_{3n})}+e^{i(\beta_{1n}+\beta_{4n})}\right)\left(\beta_{1n}^2-\beta_{4n}^2\right)\left(\beta_{2n}^2-\beta_{3n}^2\right)=0 \end{aligned} \tag{38}$$

The solution of Eqs. (36) and (38) leads to determination of the frequencies ω_n and the coefficients β_{in}. After specifying these parameters, returning to Eq. (37), the coefficients C_{in} can be determined. Then, on the basis of Eq. (35) the eigenfunctions $Y_n(x)$ are

$$\begin{aligned} Y_n(x) = c_{1n}\Bigg\{ & e^{i\beta_{1n}x} - \frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{3n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{2n}^2)(e^{i\beta_{3n}}-e^{i\beta_{2n}})} e^{i\beta_{2n}x} - \frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{2n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{3n}^2)(e^{i\beta_{2n}}-e^{i\beta_{3n}})} e^{i\beta_{3n}x} \\ & + \left(-1+\frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{3n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{2n}^2)(e^{i\beta_{3n}}-e^{i\beta_{2n}})}+\frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{2n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{3n}^2)(e^{i\beta_{2n}}-e^{i\beta_{3n}})}\right) e^{i\beta_{4n}x} \Bigg\} \end{aligned} \tag{39}$$

To receive the second order of approximation, the solution (32) is substituted into Eq. (31)

$$\begin{aligned} &\frac{\partial^2 w_1}{\partial T_0^2}+2v_0\frac{\partial^2 w_1}{\partial T_0 \partial x}+(v_0^2-1)\frac{\partial^2 w_1}{\partial x^2}+v_f^2\frac{\partial^4 w_1}{\partial x^4} \\ &= -2\frac{\partial A_n}{\partial T_1}\left(i\omega_n Y_n + v_0\frac{\partial Y_n}{\partial x}\right)e^{i\omega_n T_0} + 2\frac{\partial \overline{A}_n}{\partial T_1}\left(i\omega_n \overline{Y}_n + v_0\frac{\partial \overline{Y}_n}{\partial x}\right)e^{-i\omega_n T_0} \\ &\quad -\omega_n v_1\left[A_n\frac{\partial Y_n}{\partial x}\left(e^{i(\omega_n+\Omega)T_0}-e^{i(\omega_n-\Omega)T_0}\right)\right. \\ &\quad \left.-\overline{A}_n\frac{\partial \overline{Y}_n}{\partial x}\left(e^{i(\Omega-\omega_n)T_0}-e^{-i(\Omega+\omega_n)T_0}\right)\right] \\ &\quad +iv_0 v_1\left[A_n\frac{\partial^2 Y_n}{\partial x^2}\left(e^{i(\omega_n+\Omega)T_0}-e^{i(\omega_n-\Omega)T_0}\right)\right. \\ &\quad \left.+\overline{A}_n\frac{\partial^2 \overline{Y}_n}{\partial x^2}\left(e^{i(\Omega-\omega_n)T_0}-e^{-i(\Omega+\omega_n)T_0}\right)\right] \\ &\quad -\frac{\Omega}{2}v_1\left[A_n\frac{\partial Y_n}{\partial x}\left(e^{i(\omega_n+\Omega)T_0}+e^{i(\omega_n-\Omega)T_0}\right)\right. \\ &\quad \left.+\overline{A}_n\frac{\partial \overline{Y}_n}{\partial x}\left(e^{i(\Omega-\omega_n)T_0}+e^{-i(\Omega+\omega_n)T_0}\right)\right] \end{aligned} \tag{40}$$

The solution presented in [55] has been frequently applied and cited in later studies on dynamic of axially moving systems with time-dependent velocity (e.g., [10,56,80,81]).

The results of the publication by Öz and Pakdemirli [55] show that instabilities of the beam motion occur when the frequency of the velocity fluctuations is close to twice natural frequency of the system with constant velocity or when the frequency is close to the sum of any two natural frequencies. Principal parametric resonances and combination resonances were investigated in [55] in detail. Fig. 9 shows stable and unstable regions of principal parametric resonance for the first mode as an example of results presented in this paper. In this figure the region between the planar surfaces is unstable, whereas the remaining regions are stable. With increasing flexural stiffness value, the instability regions shift to higher Ω values. Increasing the amplitude of the velocity fluctuations enlarges the instability regions.

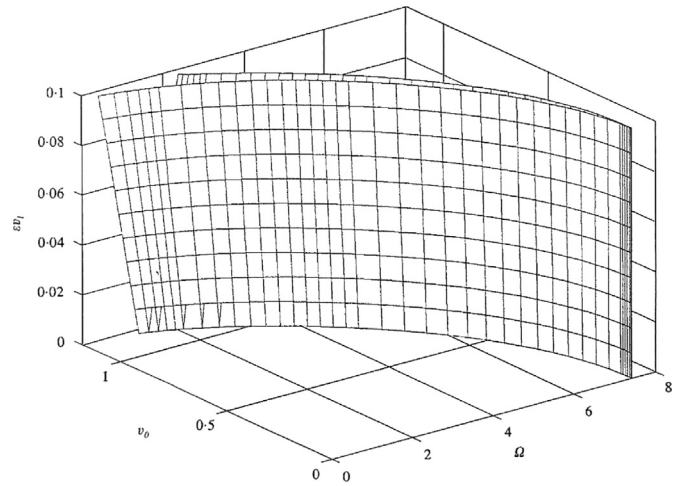

Fig. 9. Stable and unstable regions for principle parametric resonances for the first mode [55].

Sections 2 and 3 show only the most significant studies, in the authors' opinion, on dynamics of axially moving string-like and beam-like systems. The results of dynamic analysis of these systems are still published and this discussion does not exhaust all aspects of these studies. Since in this review we focus our attention primarily on the plate systems, readers interested in string-like and beam-like system are referred to the published reviews in these fields. In 1988, an extensive review of the literature showing the state of research on dynamics of axially moving string and beam systems was published by Wickert and Mote [74]. In this review the reference list includes 103 items. In 2005, a comprehensive review of studies of dynamics and control of vibration of string systems was published by Li-Qun Chen [6]. Presenting investigations that adopted linear models of moving strings, the review summarizes studies on modal analysis, complicatedly constrained strings, coupled vibrations, and parametric vibrations. Regarding investigations that adopted nonlinear models of moving strings, this review presents the governing equations with large amplitude, and reviews progress on discretized or direct approximate analytical analyses and numerical approaches based on the Galerkin method or the finite difference method. Furthermore, investigations are reported on modeling of damping mechanisms as viscoelastic materials, coupled vibration of power transmission systems, and bifurcation and chaos. The state of the art of active control of moving strings is surveyed on controllability and observability, the Laplace transform domain analysis and the energy analysis, nonlinear vibration control and adaptive vibration control are also presented. In the review [6], there are 242 references cited.

Regarding the dynamics of axially moving string-like and beam-like systems, studies are conducted till today (e.g. [9–11,15,20,21,44]). As comprehensive review of the state of researches of the beam systems as in the case of string systems has not published yet, only the reviews on distinguished systems or matters, where the beam-like model of axially moving continua was used, can be found in the literature. For instance, a comprehensive overview of researches in nonlinear dynamics of axially moving beam, there is in the paper by Pelicano and Vestroni [60] published in 2000. A comprehensive overview of state-of-studies in dynamics of pipes containing flowing fluid is located in the book by Poidousis [61], which was issued in 2004. In 2008, the state of researches in the dynamics of the axially moving beam with variable speed in the paper by Pakdemirli and Öz [58] was presented.

4. Dynamics of axially moving elastic plates

As follows from the description presented in the previous sections, to avoid too complex a mathematical description, the string or beam theory was used in numerous earlier studies to model axially moving two-dimensional systems. Although this simplification leads in many cases to satisfactory results, however it cannot be applied in numerous instances. Such situations occur when the plate width and its orthotropic properties, a change in the distribution of load along the plate width, a lack of free ends of the plate, composite plates or intermediate linear or point supports of the plate are taken into consideration. In all these instances, it is necessary to employ the two-dimensional plate theory in dynamic investigations.

The first paper in which a dynamic analysis of the plate model of a wide blade of the band saw was analyzed was authored by Ulsoy and Mote [69]. The model of the plate moving with a constant axial speed was analyzed on the assumption of typical membrane stresses' effect. The differential equation of motion with partial derivatives was discretized with the Ritz method. Approximate boundary conditions on free edges of the plate were taken into consideration, which – as pointed out in further studies – affects fundamentally an accuracy of the solution, especially in the overcritical range of the plate motion [31,32].

Later studies devoted to investigations of the dynamics of moving plate systems were issued in the 1990s. Owing to the regions of industrial applications, these investigations were focused mainly on analyses of the dynamics of the moving web, that is to say, a thin plate characterized by low flexural stiffness. In 1995 Lin and Mote [35] published their paper, in which – starting with the Karman theory – the mathematical model that describes the transverse motion and the field of cross-sectional forces of the axially moving isotropic web is determined. This model was derived under assumptions that the propagation of elastic waves in the web plane is not taken into account and the kinetostatic effects are neglected. Then the mathematical model has a form of the system of two coupled nonlinear differential equations with partial derivatives:

$$\rho(-w_{,tt}-2\,v\,w_{,xt}-v^2 w_{,xx})-F_{,yy}w_{,xx}-F_{,xx}w_{,yy}+2\,F_{,xy}w_{,xy} \\ +D\,(w_{,xxxx}+2\,w_{,xxyy}+w_{,yyyy})=P, \tag{41}$$

$$F_{,xxxx}+2\,F_{,xxyy}+F_{,yyyyy}=E\,h\,(w_{,xy}^2-w_{,xx}\,w_{,yy}), \tag{42}$$

where ρ is the mass density, h is the thickness of the web, w is the transverse displacement, v is the transport speed, P is the transverse loading, D is the plate stiffness, F is the the stress function.

In the study by Lin and Mote [35] the results of investigations of transverse displacements and a distribution of stresses under the transverse loading were presented. In this publication, an influence of boundary regions of the web on displacements and a stress distribution in the equilibrium state was also investigated.

Dynamic stability of a moving rectangular plate subject to in-plane acceleration and force perturbations is studied in the paper [30]. In 1997 the fundamental study showing vibration characteristics and results of investigations of stability of axially moving plates within the linear theory was published by Lin [34]. The vibrations of two-dimensional, axially moving isotropic plate with freely supported ends and two free ends were analyzed. A constant tension force along the transport speed acted on the freely supported ends in the plane of the plate. The boundaries of instability regions of the plate motion were determined in two ways: through the determination of existence of the nontrivial equilibrium position using a static analysis and through an examination of eigenvalues of the discretized system using a dynamic method. The plate transport speed corresponding to the stability boundary (critical speed) was determined as the lowest transport speed at which there existed a nontrivial equilibrium position of the plate or the lowest speed at which the real part of one of the eigenvalues became nonzero. The results of the investigations conducted by Lin prove that both the velocities coincide.

Lin proved that the critical value of the plate transport speed, similarly as in the case of the string, was equal to the speed of transverse wave propagation in the plate. The value of the critical speed increases if the flexural stiffness of the plate increases and the length to width ratio (slenderness) of the plate decreases. The results of the conducted investigations show as well that the one-dimensional theory of beam always overestimates the value of the plate critical speed, whereas the string theory always underestimates this speed. This regularity demonstrated in [34] is illustrated in Fig. 10. For various slendernesses of the plate ($\xi=l/b$), a relationship between the dimensionless critical speed C^* and the dimensionless plate stiffness ε has been plotted in this figure.

The complexity of mathematical model of axially moving plate caused that many researchers applied the finite element method in the studies of its dynamics. On the basis of the Mindlin–Reissner plate theory, Wang in [71] presented a description of the finite element for an axially moving thin plate. It was found that it was possible to determine not only the critical transport speed of the web but a distribution of normal stresses and shear stresses in the moving web with such finite elements. With the developed finite elements, numerical calculations of eigenfrequencies and critical speeds of the axially moving paper web were carried out. The description presented in [71] was improved and developed later by Laukkanen in the paper [29].

In 2002 Luo and Hutton in the study [37] presented a formulation of the moving triangular isotropic plate finite element loaded with membrane and gyroscopic forces. The mathematical model of this element was derived through a modification of the Kirhchoff discrete theory. Some exemplary results of dynamic calculations of the axially moving plate under loading with a variable distribution presented in this study were verified by a comparison to the results of calculations obtained with the Releigh–Ritz method. Another formulation of the finite element was presented in [26], where equations of the modal finite element within the frequency domain were formulated on the basis of Kantorowitz' method. This element was then applied to dynamic calculations of the axially moving plate. The results of these calculations were compared in [26] to the analytical solution and to the FEM solution. A new shell element in a finite element procedure is used in [27,28] to study the dynamic behavior of a paper web. Effects of transport velocity and surrounding air are taken into consideration in this analysis.

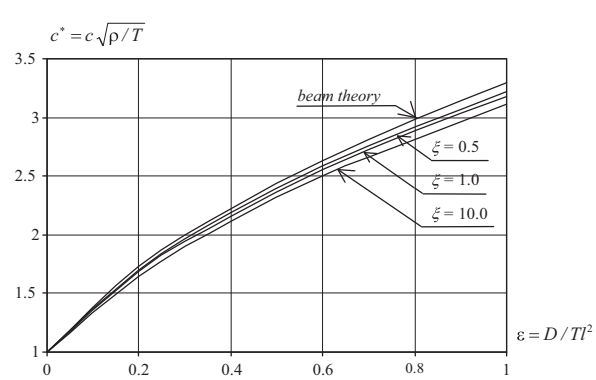

Fig. 10. The critical transport speed of the plate for various plate stiffness and slenderness ratio [34].

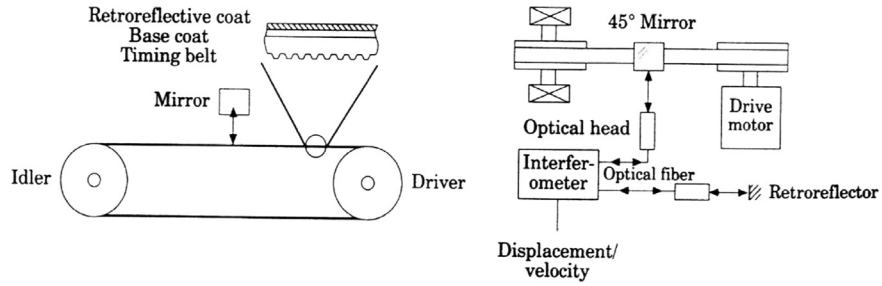

Fig. 11. Schematic of the test used to measure the vibration of power transmission belts [49].

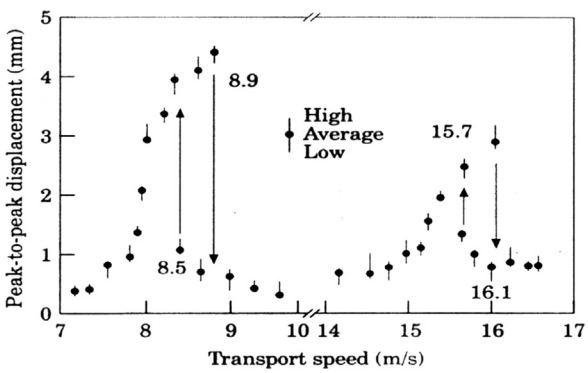

Fig. 12. Measured speed-dependent belt response amplitudes over resonance of the first two vibration modes. Data in the non-resonance range 10–14 m/s are omitted for clarity [49].

Among many references published in the late 1990s and the early 2000s which are devoted to the dynamic investigations of axially moving systems, special attention should be drawn to the publication by Moon and Wickert [49] and the publication by Pellicano et al. [59] both devoted to analysis of nonlinear forced vibrations of power transmission belts. These are the only publication to the knowledge of the authors of the present paper in which the results of theoretical nonlinear analyses of axially moving system have been compared to the experimental measurement results. The same theoretical nonlinear model of the axially moving beam proposed in the publication by Wickert [72] has been used in both investigations.

The most significant experimental results obtained by the Moon and Wickert are shown below. The experimental investigations included in [49] were carried out on the laboratory test stand, whose general view is presented in Fig. 11.

A seamless timing belt with a span of $L=860$ mm was driven by a variable speed rotor over drive and idle pulleys of a common pitch radius equal to $r=63$ mm. Nonlinear vibrations of the laboratory belt system were excited by pulleys having slight eccentricity. An idle pulley was mounted on rotation and translation edges in order to minimize those excitation sources that were associated with the pulley misalignment.

A Michelson-type laser interferometer was used to measure transverse vibrations of the belt. Reference paths and target light beams were established with fiber optic leads. The interferometer measured alternations in lengths of those two light paths through the interference fringes generated by superposition of coherent beams that reflected from a stationary reference surface and a moving target, that is to say, the belt itself. The belt was coated with a thin layer of retroreflective paint to ensure that sufficient light was scattered from the belt and into the optical head. Particles within the paint matrix ensured that a portion of the incident light was returned into the source optical fiber regardless of the belt finite amplitude or slope. Owing to this technique, vibration measurements were made with a strong signal-to-noise ratio. A displacement resolution (sub-micron) and bandwidth (DC to 100 kHz) exceeded the test requirements.

The eccentricity of each pulley with respect to its shaft rotation axis was recorded with an eddy current probe, whose linear range was of approx. 2 mm. Measurements were made on top lands of the pulley teeth. Such eccentricity may follow from manufacturing tolerances (especially in the case of cast components) and a clearance between the pulley hub and the shaft necessary for assembly. The measured profiles indicated peak-to-peak run-out levels for the drive and idle pulleys of 0.71 and 0.42, correspondingly. The maxima were shifted in phase by approx. 240° of rotation. The relative phase ϕ can be nulled or adjusted to an arbitrary value by proper synchronization of the pulleys, but under the test conditions, ϕ will generally be nonzero.

Fig. 12 presents measurements of the belt peak-to-peak response amplitude versus the running speed over $0<V<17$ m/s. This range encompasses resonance of the first two modes. The speed was increased monotonically and it was then held constant. Each amplitude measurement was taken about 1 cm from the driven pulley. The time records of transient response shown in Fig. 13, obtained as the speed either increased or decreased through resonance of the fundamental mode near 9 m/s, are associated with the amplitude in Fig. 12. When the running speed was increased from 8.4 to 9.2 m/s in Fig. 13a, the amplitude decreased sharply from the maximal value of 1.98 mm to 0.57 mm near $V=8.9$ m/s, as in Fig. 12. In the alternative case of Fig. 13b, when the speed was instead decreased through the same range, the amplitude grew from 0.52 to 1.51 during passage through $V=8.5$ m/s.

Similar non-reversible behavior, in which the vibration amplitude was dual values and path dependent, was observed at speeds 15.7 m/s and 16.1 m/s, and corresponds to resonance of the belt second mode (Fig. 12). Such classic "jump" behavior is characteristic of structural nonlinearity, and it is attributed here to the longitudinal stretching resulting from a finite amplitude motion in the near-resonance and resonance region.

Among works published in the 1990s and 2000s which are devoted to axially moving web control is worth to note the studies of Young et al. [78,79], a practical study of web position in wallpaper machine [70] and the study of transverse vibration control of an axially accelerating web [54]. In these works, beside dynamic analysis, methods to control both lateral and longitudinal motions of the web are included. In addition, modeling and control of multiple web spans is studied in [79]. Using the date from the real wallpaper machine, the mathematical model is obtained and validated against the real system response in [70]. To facilitate the Lyapunov analysis for the vibration regulation in [54], a control strategy for an axially moving web system is presented.

Dynamic characteristics of the out-of-plane vibration for an axially moving membrane are studied by Shin et al. in [64]. An extensive study by Luo and Hamidzadech [36] was issued, where equilibrium, membrane forces and buckling stability of thin plates

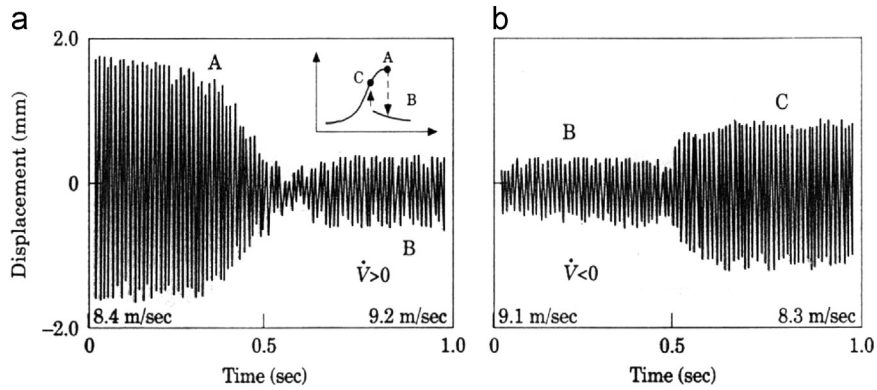

Fig. 13. Measured time records depict jumps in amplitude over the first resonance region for monotonically (a) increasing and (b) decreasing running speeds [49].

were analyzed analytically. Within the Karman linear theory of plates and the nonlinear membrane theory, the equations of motion of the axially moving orthotropic plate were derived. Solutions to the motion equations were obtained with a perturbation method. Numerical computations were conducted for simple supports of all plate ends and for constant longitudinal and transverse vibrations. The authors of [36] have formulated an important conclusion that while analyzing high modes of the plate deflection, the nonlinear plate theory should be the basis for analysis even if amplitudes of the deflection are very low compared to the plate thickness.

Recently, vibrations and stability of an axially moving rectangular antisymmetric cross-ply composite plate were studied by Yang et al. [77]. The partial differential equations governing the in-plane and out-of-plane displacements are derived by the balance of linear momentum. The natural frequencies for the in-plane and out-of-plane vibrations are calculated by both the Galerkin method and the differential quadrature method. Investigation results show that natural frequencies of the in-plane vibrations are much higher than those in the out-of-plane case, which makes considering out-of-plane vibrations are only reasonable. The instability caused by divergence and flutter is discussed by studying the complex natural frequencies for constant axial moving velocity. For the axially accelerating composite plate, the principal parametric and combination resonances are investigated by the method of multiple scales. Also nonlinear free transverse vibrations of moving plates without and with internal resonances were studied by Tang and Chen [67].

The loss of stability of axially moving elastic plate was investigated also in the study by Banichuk et al. [3]. Analysis was performed in an analytical manner for static modes of instability. The critical divergence velocity and the corresponding buckling shapes were studied as functions of geometric and mechanical parameters. It was shown that the buckled plate shape is symmetrical and the antisymmetric shapes correspond to higher values of the transport speed. Also it was shown that the meaningful elastic deformation becomes localized at the vicinity of the edges of the plate, and the amount of localization only depends on the Poisson and aspect ratios of the plate.

5. Dynamics of axially moving viscoelastic plates

An important problem that can be met while considering the dynamic behavior of the axially moving two-dimensional system is how to model the plate material. In many instances the simplest elastic model is not sufficient and moving continua have to be considered as viscoelastic. It makes the dynamic analysis much more complicated. In the case of viscoelastic materials, the state of strain in an element at a particular time not only depends on the state of stress at this time, but also on the history of stresses. Similarly, the state of stress of an element depends on the history of strains. This memory effect should be taken into account in dynamic investigations.

The literature especially related to transverse vibrations of viscoelastic plates is rather limited. In the case of a stationary orthotropic plate, its two-dimensional rheological model was described fairly long ago by Sobotka in [66]. A two-dimensional rheological element was applied in dynamic investigations of axially moving systems by one of the co-authors in the paper [41]. In the literature, one can find also works in which one-dimensional rheological models were used to describe properties of the axially moving material. The paper by Fung et al. [17] was the first one where transverse vibrations of a viscoelastic moving belt represented by the spring model were investigated. Nonlinear dynamics of an axially accelerating viscoelastic beam model was investigated by Chen et al. [8] and Yang and Chen [76]. In those papers, the moving material was described with the Kelvin–Voigt rheological model. Regular and chaotic vibrations of an axially moving viscoelastic beam model of the web subjected to tension variations were studied in Marynowski and Kapitaniak [45,46] and Marynowski [42]. In those works three different rheological models, namely a two-parameter Kelvin–Voigt model a three-parameter Zener model, and a four-parameter Burger model were applied. Comparative analysis of the results published in these works is presented in Section 6.

In 2007 Zhou and Wang [82] studied vibration characteristics of an axially moving viscoelastic rectangular plate. Authors assumed the plate to be elastic in dilatation and viscoelastic in distortion, where the viscoelasticity was described with the Kelvin–Voigt model. The same authors continued their studies on transverse vibration characteristics of moving viscoelastic plates taking into account a parabolically plate thickness [83]. Also in that time two studies by Hatami et al. [22,23] devoted to free vibrations of axially moving multi-span composite plates and a viscoelastic Navier-type plate were published. In those papers, an exact finite strip method was developed for dynamic analysis of axially moving elastic and viscoelastic plates. The exact stiffness matrix of a finite strip of plate is defined in the frequency domain. By joining these matrices, the global stiffness matrix of the whole plate is obtained. In the case of the viscoelastic plate with all simply supported edges, the eigenvalues of the global stiffness matrix have the form of complex numbers.

Recently, Tang and Chen in the paper [68], and Yang et al. [75], studied vibrations, bifurcations and chaos of axially moving viscoelastic plates using finite difference and nonlinear model for transverse displacements. They paid attention on bifurcations and chaos but also studied the dynamic characteristics of a linearized elastic model with the help of eigenfrequency analysis.

6. Review of selected works published by the authors

The investigation results published by the authors in the four papers [41,43,45,46] are presented in this section. All these publications present the results of dynamic analysis of axially moving viscoelastic web. In individual studies different rheological models of the web material were used. Since three tests were carried out on the same object, comparative analysis allows us to investigate the effect of modeling of the web material on dynamic behavior of the web in both undercritical and supercritical range of transport speed.

6.1. Modeling of the web

In the presented papers, published over 8 years, different methods were used to study the dynamics of axially moving continua. The web was treated as a viscoelastic thin plate, moves along the longitudinal direction x at the constant velocity c. The co-ordinate system and the geometry of the system are shown in Fig. 14. The forces that act on a small element of the plate dA are shown in Fig. 14 as well.

As shown in previous sections, in modeling the axially moving webs one can use one-dimensional beam theory or two-dimensional plate theory. Both these theories have been applied in the presented studies. In the earlier works [45,46] the beam models of the web were tested, and in the works [41,43] the plate model was used. Three different one-dimensional rheological models, namely the two-parameter Kelvin–Voigt model, the three-parameter Zener model, and the four-parameter Bürgers model were used in the studies [43,45,46] to model the viscoelastic material of the web (Fig. 15).

In the study [41] and also in the book [42] a two-dimensional rheological model for orthotropic viscoelastic material is used in dynamic analysis of moving system. This model is presented in Fig. 16.

To formulate the constitutive equation of this two-dimensional model, the generalized Hook's law and the creep functions of the web material along the main directions of orthotropy have been employed. Rheological properties of the web along both directions have been described with the one-dimensional three-parameter Zener model. The creep functions of the Zener rheological element (Fig. 15) have the following form:

$$\phi(t)=\frac{1}{E_1}-\frac{E_2}{E_0E_1}e^{-(E_{1x}/E_{0x}\lambda)t}, \tag{43}$$

where $E_0=E_1+E_2$, $\lambda=\eta/E_2$.

To determine the shear stress, the relaxation function of the Zener-type viscoelastic element has been taken into consideration in [41]

$$\psi_{xy}(t)=G_1+G_2e^{-t/T_G}, \tag{44}$$

where $G=G_1+G_2$ is the shear modulus of the material and T_G is the relaxation time constant.

Mathematical model in general form of partial differential equation, resulting from Hamilton's principle, for a transverse motion of the two-dimensional axially moving viscoelastic plate, was derived earlier by one of the co-authors [40,42] and has the following form:

$$\rho h\left(\frac{\partial^2 w}{\partial t^2}+2c\frac{\partial^2 w}{\partial x\,\partial t}+c^2\frac{\partial^2 w}{\partial x^2}\right)-\frac{\partial(N_x w_{,x})}{\partial x}-\frac{\partial(N_y w_{,y})}{\partial y}-\frac{\partial(N_{xy} w_{,x})}{\partial y}-\frac{\partial(N_{xy} w_{,y})}{\partial x}-\frac{\partial^2 M_x}{\partial x^2}-2\frac{\partial^2 M_{xy}}{\partial x\partial y}-\frac{\partial^2 M_y}{\partial y^2}=0, \tag{45}$$

where ρ is the mass density; h is the thickness; w is the transverse displacement of the plate; M_x, M_y, M_{xy} are the bending moments per unit length; N_x, N_y, N_{xy} are the in-plane forces per unit length.

In all presented studies Eq. (45) was the basis for determining the governing equation of motion of the test systems. In the works [45,46] geometric nonlinearity has been taken into account. Then the nonlinear strain component in the longitudinal direction x is

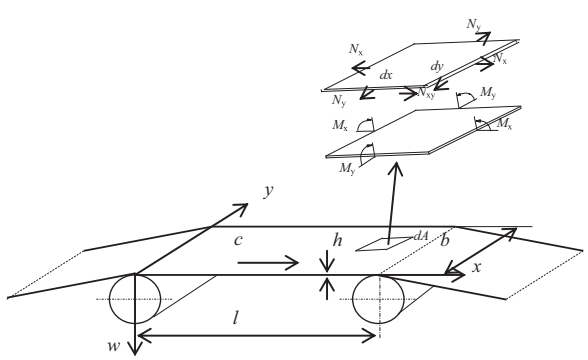

Fig. 14. Axially moving plate and internal forces act on an infinitely small element dA.

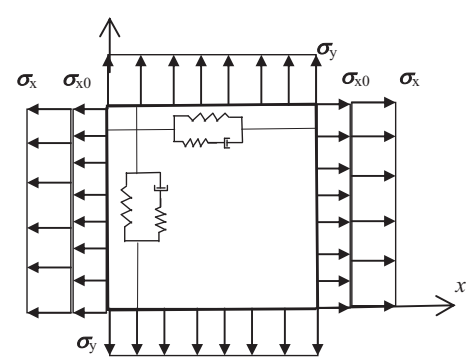

Fig. 16. Two-dimensional rheological model.

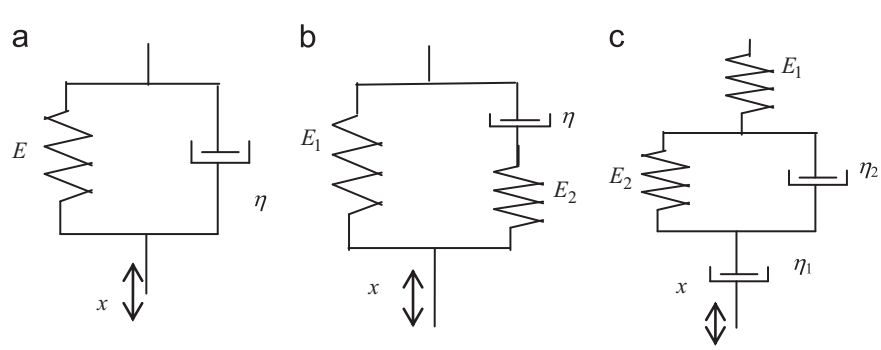

Fig. 15. Rheological models used in the authors' studies: (a) Kelvin–Voigt model, (b) Zener model and (c) and Bürgers model.

related to the transverse displacement w by

$$\varepsilon(x,t)=\frac{1}{2}w_x^2(x,t). \tag{46}$$

In addition to the work [41], where the two-dimensional model was used, to model the web material in the study [45] the one-dimensional Kelvin–Voigt model and the Bürgers model were used. In the work [46] the Zener rheological model, and in the work [43] the Kelvin–Voigt model and the Zener model were applied. Only in the work [46] the tension of the web was characterized as a periodic perturbation on the steady-state tension, in the remaining works the constant web tension was taken into account.

In the papers [45,46] to derive the equations of motion of the web corresponding constitutive equations were included in Eq. (46). Then, the received relationship was substituted into the one-dimensional form of Eq. (45). In this way, in the studies [45,46] the beam models of axially moving web were designated. The simple boundary conditions are taken into consideration. The problems represented by the governing equations of the beam models together with boundary conditions have been solved using the Galerkin method. The 4-term finite series representation of the dimensionless transverse displacement of the beam has been taken in numerical investigations in the time domain.

A different research method was used in [43]. In this work, the plate model of viscoelastic web was studied in the frequency domain. Taking into account elastic–viscoelastic equivalence and the equilibrium state equation in the complex frequency domain, a simple calculation method for dynamic analysis of the two-dimensional axially moving viscoelastic web was proposed in the study [43]. In this paper equivalent of Boltzmann's superposition principle in the complex frequency domain is presented in the following form:

$$\sigma(\omega)=R(\omega)\,\varepsilon\,(\omega), \tag{47}$$

where $R(\omega)$ is the Laplace transform of the relaxation function in the frequency domain, $\sigma(\omega)$ is the Laplace transform of the stress function, $\varepsilon(\omega)$ is the Laplace transform of the strain function.

For isotropic plate the deviation tensor's components of stress s_{ij} and strain e_{ij} are introduced and then Eq. (47) is reduced to two independent equations:

$$s_{ij}(\omega)=2G(\omega)e_{ij}(\omega),\ \sigma_{kk}(\omega)=3\kappa(\omega)e_{kk}(\omega), \tag{48}$$

where

$$s_{ij}=\sigma_{ij}-\frac{1}{3}\delta_{ij}\sigma_{kk};\quad s_{ii}=0,$$
$$e_{ij}=\varepsilon_{ij}-\frac{1}{3}\delta_{ij}\varepsilon_{kk};\quad e_{ii}=0, \tag{49}$$

$G(\omega)$ is the complex shear modulus, $\kappa(\omega)$ is the complex bulk modulus, δ_{ij} is the Kronecker's delta.

$G(\omega)$ represents the behavior of viscoelastic material under simple shear and $\kappa(\omega)$ under hydrostatic stresses. Each of these modules can be defined independently in the frequency domain, using one of the rheological models.

To model the behavior of the moving web under shear stresses, a three-parameter Zener rheological model (Fig. 15) was used in the paper [43], whereas the hydroelastic behavior was considered as elastic. The constitutive equation of the Zener rheological model has the following form:

$$\eta\frac{d\,\sigma}{d\,t}+G_2\sigma=\eta\,(G_1+G_2)\frac{d\,\varepsilon}{d\,t}+G_1G_2\varepsilon. \tag{50}$$

Taking into account the relaxation time constants:

$$T=\frac{\eta}{G_2};\quad T_1=\frac{\eta}{G_1}:\quad T_0=T+T_1, \tag{51}$$

the constitutive equation has the following form:

$$T\frac{d\,\sigma}{d\,t}+\sigma=G_1\left(T_0\frac{d\,\varepsilon}{d\,t}+\varepsilon\right). \tag{52}$$

In the free vibration problem, the stress and strain functions can be expressed as:

$$\sigma=\sigma_0\exp(i\,\omega\,t);\quad \varepsilon=\varepsilon_0\exp(i\,\omega\,t). \tag{53}$$

Substituting Eq. (53) into the constitutive Eq. (52), one receives the shear modulus in the frequency domain

$$G(\omega)=\frac{\sigma_0}{\varepsilon_0}=G_1\frac{1+i\,\omega\,T_0}{1+i\,\omega\,T}. \tag{54}$$

It is worth noticing that in the case $T_0=0$, Eq. (54) defines the shear modulus for the two-parameter Kelvin–Voigt rheological model, and for $T_0=T$ is the shear modulus in the elastic case. Taking into account Eqs. (49) and (54), the engineering modules take the following forms:

$$E(\omega)=\frac{9\,\kappa_0G(\omega)}{3\,\kappa_0+G(\omega)}=\frac{9\,\gamma\,G_1(1+i\omega\,T_0)}{3\,\gamma\,(1+i\omega\,T)+1+i\omega\,T_0}, \tag{55}$$

$$\nu(\omega)=\frac{3\,\kappa_0-2\,G(\omega)}{2\,(3\,\kappa_0+G(\omega))}=\frac{1}{2}\frac{3\,\gamma\,(1+i\omega\,T)-2\,(1+i\omega\,T_0)}{3\,\gamma\,(1+i\omega\,T)+1+i\omega\,T_0}, \tag{56}$$

where κ_0 is the bulk modulus at the zero frequency

$\gamma=\kappa_0/G_1$.

The equilibrium state of the axially moving web is defined as the nontrivial equilibrium position when time-dependent forces do not interact on the web and the form of deflection is dependent only on the transport speed. After removing the time-dependent components in Eq. (45) the equation of equilibrium states of the axially moving viscoelastic web which is tensioned with the constant longitudinal load N_{x0} has the following form:

$$\rho\,h\,c^2\frac{\partial^2 w}{\partial\,x^2}-N_{x0}\frac{\partial^2 w}{\partial\,x^2}+D(\omega)\left(\frac{\partial^4 w}{\partial\,x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial\,y^4}\right)=0, \tag{57}$$

where ρ is the mass density; h is the thickness, w is the transverse displacement, c is the transport speed

$$D(\omega)=\frac{h^3}{12}\frac{E(\omega)}{(1-\nu(\omega)^2)}. \tag{58}$$

In the paper [43] the boundary conditions referring to simply support at transverse ends and longitudinal free ends (Fig. 14). In all the reported works derived mathematical models are presented in dimensionless forms.

6.2. Results of numerical investigations

In three presented studies, [43,45,46], numerical investigations have been carried out for the axially moving steel web of the length $l=1$ m, and the thickness $h=0.0015$ m. Mass density of the web $\rho=7800$ kg/m^3 and Young's modulus along the longitudinal direction: $E_x=0.2\times10^{12}$ N/m^2. The phase portraits, Poincare maps and bifurcation diagrams were used to analyze nonlinear systems in the reported works [45,46]. The bifurcations diagrams ware presented by varying the dimensionless parameters: transport speed s and the internal damping coefficients β in both works. The amplitude of the tension periodic perturbation α played the role of the bifurcation parameter only in the work [46].

The analysis of the linearized beam models with Kelvin–Voigt rheological element and with Bürgers element shows that in the subcritical range of transport speed the increase of this speed causes the decrease of the frequency of the natural oscillations. For small values of internal damping coefficient β at critical transport speed the system exhibits the divergent instability. For super-critical transport speeds and small internal damping the web

experiences divergent and flutter instabilities. Between these two instability regions there is the second stability area. The width of this region depends on the internal damping of the web material. When the internal damping increases the width of the second stable region decreases more and finally disappears. Fig. 17 shows the map of stability and instability regions calculated for the linearized beam models with Bürgers rheological element.

For the larger values of internal damping ($\beta > 3 \times 10^{-5}$) the beam models compared in [45] behave differently. The linearized beam model with Bürgers element loses its stability due to the flutter instability. This is the significant difference between both considered models, as the beam model with Kelvin–Voigt element does not allow the identification of the first flutter instability region shown in the map in Fig. 17. The critical value of the transport speed of the beam model with Kelvin–Voigt element occurs when the system reaches the divergence instability.

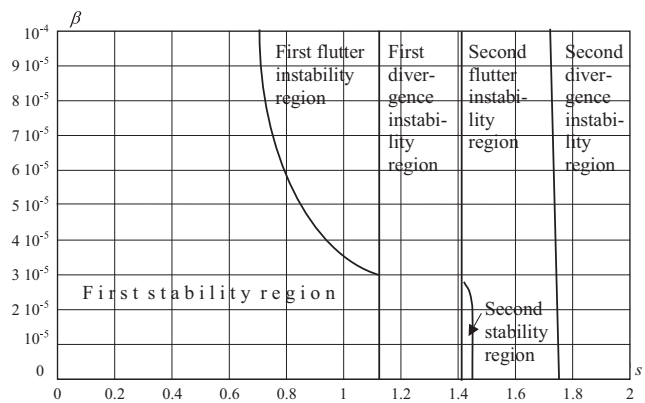

Fig. 17. Instabilities regions of the linearized beam model with Bürgers element [45].

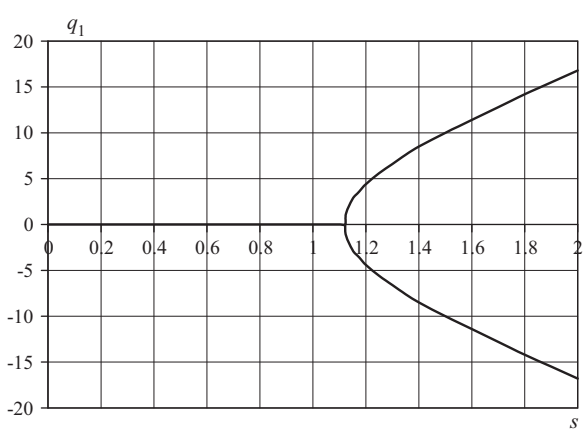

Fig. 18. Bifurcation diagram of the nonlinear system with Kelvin–Voigt element of axially moving material ($\beta = 10^{-5}$) [45].

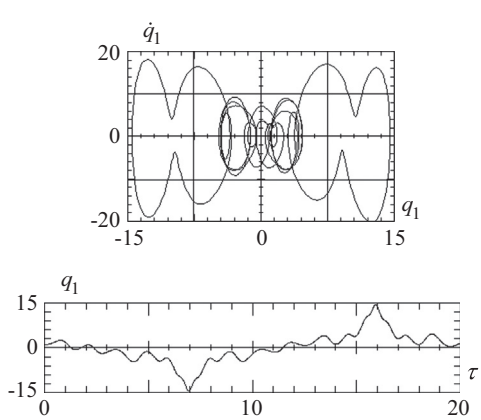

Fig. 19. The phase portrait (a) and the time history of the nonlinear system with Kelvin–Voigt element of axially moving material ($s = 1.54$; $\beta = 0$) [45].

The dynamic analysis of the nonlinear damped beam models shows in supercritical transport speed region the nontrivial equilibrium positions bifurcate from the straight configuration of the web and global motion between the co-existing equilibrium positions occurs. At critical the transport speed $s_{cr} = 1.12$ the pitchfork type bifurcation occurs (Fig. 18). The phase portrait in Fig. 19 shows above the critical transport speed for different values of the internal damping, the nonlinear system may reach various equilibrium positions.

Similar dynamic behavior like the beam model with Bürgers element has been shown in the study [46] investigating the beam model with Zener rheological element. The analysis of the linearized system predicts the flutter instability and exponentially growing oscillations in supercritical region of transport speed. The critical value of the transport speed increases with the increase of damping coefficient β. The dynamic behavior of the parametrically unexcited nonlinear beam model with Zener element shows the bifurcation diagram of the dimensionless displacement of the web v in Fig. 20. The dimensionless transport speed s has been used as the bifurcation parameter.

In Fig. 20 one can observe supercritical Hopf-type bifurcation at the transport speed $s_{cr} = 0.71$. It is worth to note that in the previous beam model with Bürgers element, the transport speed $s = 0.7$ has been identified as the critical transport speed for the considered damping coefficient value. Though the analysis of the linearized system predicts exponentially growing oscillations for $s > s_{cr}$,

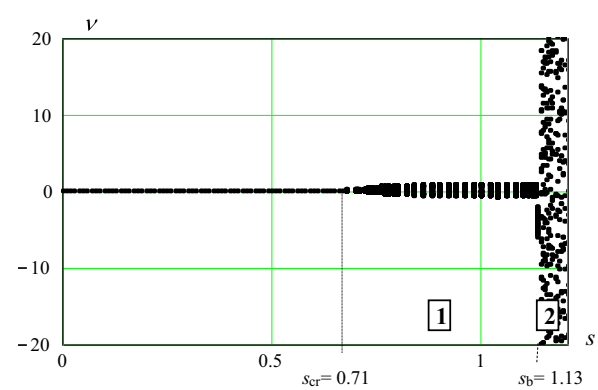

Fig. 20. Bifurcation diagram; $\beta = 1.778 \times 10^{-4}$, $\omega_0 = 3.5$, $\alpha = 0$ [46].

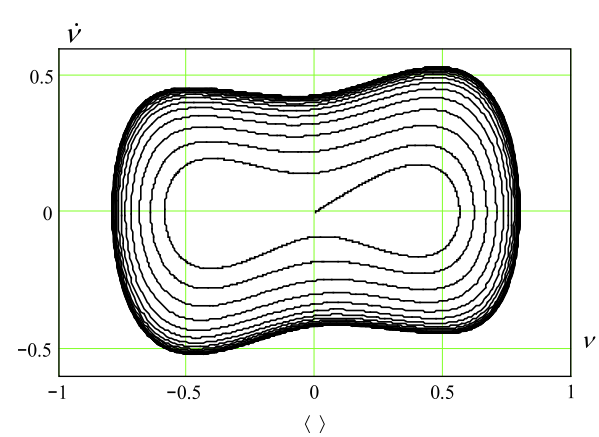

Fig. 21. The phase portrait of the solution of nonlinear system: $s = 1.15$, $\beta = 10^{-3}$, $\alpha = 0$, $\omega_0 = 3.5$ [46].

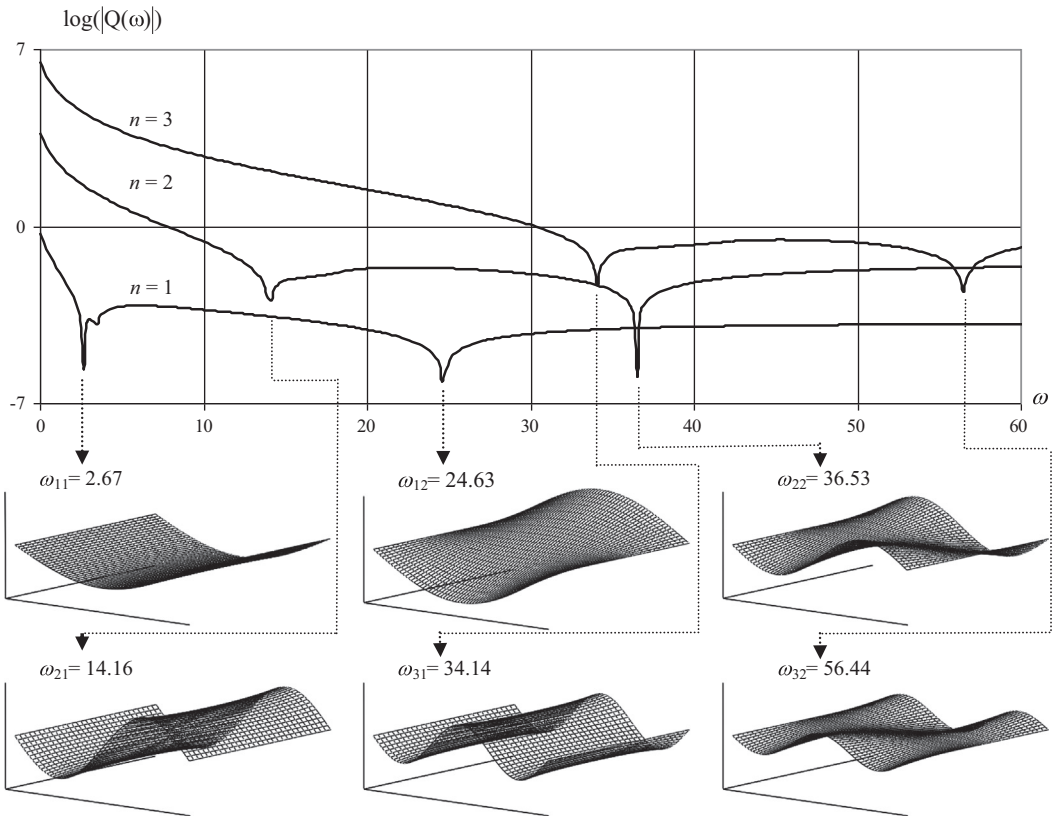

Fig. 22. Frequencies and deflection forms at equilibrium positions (Kelvin–Voigt rheological model, $s=1.03$, $\tau=1.46$) [43].

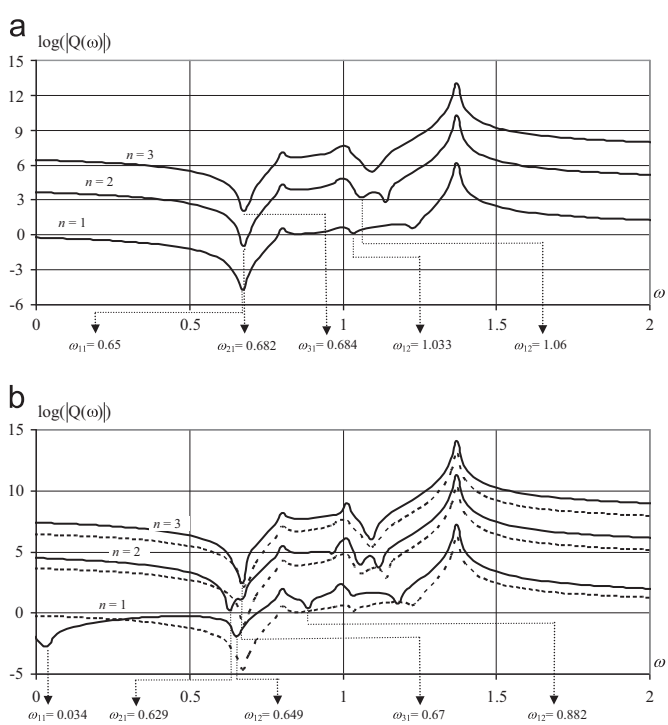

Fig. 23. Frequencies of successive modes (Zener rheological model, $\tau=0.73$, $\tau_0=1.46$): (a) $s=1.03$ and (b) $s=1.03$ and 1.26 [43].

nonlinear damped oscillations which tend to the stable limit cycle motion occur (region 1 in Fig. 20). If the transport speed is increased further at $s_b=1.13$ the second bifurcation occurs. At the transport speeds above the bifurcation point (region 2 in Fig. 20), the parametrically unexcited nonlinear system exhibits global motion between two center points (Fig. 21). It is worth to note that in the previous beam model with Kelvin–Voigt rheological element, the transport speed $s=1.12$ has been identified as the critical transport velocity when the pitchfork type bifurcation occurs (Fig. 18). In the beam model with Zener rheological element above this transport speed one can observe the coexistence of attractors.

As is already known, the linear plate model derived in the complex frequency domain in the work [43] allowed the identification of equilibrium states of the moving web. The equilibrium states are defined as the nontrivial equilibrium positions of the moving web in the range of supercritical transport speeds. The investigation results in the form of a plot of the decimal logarithm of the absolute value of the characteristic determinant $Q(\omega)$ of the investigated system versus the frequency value for the dimensionless transport speed $s=1.03$ and the time constant of the Kelvin–Voigt model $\tau=1.46$ are depicted in Fig. 22. The local minima on three curves in Fig. 22 determine natural frequencies of six successive modes of vibrations. The deflection forms of the axially moving plate in equilibrium states which are connected with successive natural frequencies are plotted in Fig. 22.

The investigation results in an analogical form of the plot of the decimal logarithm of the absolute value of the characteristic determinant of the system with Zener rheological element versus the frequency value are shown in Fig. 23. In Fig. 23a the local minima on three curves determine natural frequencies of the steel plate for $s=1.03$ and the relaxation dimensionless time constants of the Zener model $\tau=0.73$ and $\tau_0=1.46$. The same plot is repeated in Fig. 23b with a dashed line for $s=1.26$. A comparison of the curves in Fig. 23 shows that when the transport speed increases, the sequence of modes changes in the plate modeled with the Zener rheological element. The modes with a one-half wave along the longitudinal direction appear earlier than at a lower transport speed. The critical transport speed has a greater value than for the Kelvin–Voigt rheological model.

The investigation results published by the authors in the papers [43,45,46] show that a particular rheological model of the moving viscoelastic material should be used in dynamic analyses depending on the nature and the purpose of research. Numerical studies of the beam models with Kelvin–Voigt and Bürgers elements show that both models give similar results for small values of internal damping. The model of axially moving material with two-parameter Kelvin–Voigt rheological element is much less complicated and can be successfully used to describe the dynamic behavior of linear models of axially moving web in undercritical range of transport speeds.

Different conclusions follow the test of the dynamics of nonlinear model of moving web with supercritical speeds. Comparing the Kelvin–Voigt model and the Zener model in studies of nonlinear dynamics of the web in the supercritical range of velocities, more accurate results are obtained using three-parameter Zener rheological element. Performed in [43,46] studies the dynamic behavior of the web moving with supercritical velocities using the beam model and the plate model confirm suitability of this method of modeling of axially moving material.

7. Concluding remarks and directions for further studies

The up-to-date survey of the knowledge on dynamic analyses of axially moving continua in this paper allows for stating that this subject belongs to one of the most currently developed in the last 60 years. In the early stage, mainly string-like and beam-like systems were the object of interest for scientists. The dynamic behavior of plate-like systems has been the least recognized during long time. Since in the last time many studies in dynamics of axially moving plate-like systems were published in this review the authors decided to present and reorder them. Unlike the string-like and beam-like systems, such review on the plate-like systems has not been published yet.

The mathematical models, derived in the studies of transverse vibrations of the axially moving plates, include the most important factors that exert an influence on the system dynamic behavior. The following can be mentioned among these factors: transport speed, the plate geometry, magnitude and distribution of axial stresses, viscoelastic properties of the plate material and nonlinear geometrical relationships between plate strains and displacements. This last factor has caused that the derived mathematical model takes a form of a system of coupled nonlinear differential equations with partial derivatives. Because it is impossible to find an exact solution to this system of equations, its approximate solution is determined using varies methods, for instance the Galerkin method, the finite difference method or the differential quadrature method. Also the complexity of mathematical model of axially moving plate caused that many researchers applied the finite element method in the studies of its dynamics.

This review of literature on dynamics of the axially moving materials allows for determination of further directions of dynamic investigations in this field. In the authors opinion, analyses should be extended by investigations comprising nonlinearity in the two-dimensional model. It concerns both the geometrical nonlinearity, as well as that one which follows from the physical properties of the moving material. The results obtained so far show that the effect of nonlinearity on the behavior of the plate obtained during the investigations is very strong indeed. Previous studies in this field have shown that, for example, the nonlinear plate theory should be the basis for analysis even if amplitudes of the deflection are very low compared to the plate thickness.

In future studies to describe the rheological properties of the moving material more accurate rheological models should be introduced. When the classical rheological models such as, for example, the Kelvin–Voigt model, or Zener model are taken into considerations, the calculations are relatively simple, but their compatibility with the empiric observations is usually poor. A better approximation can be achieved using fractional models. In these models the stress–strain relation is represented in terms of non-integer order derivatives [62]. A four-parameter fractional derivative model, also known as the generalized Zener model, has been used in dynamic studies of stationary systems (e.g. [25,39]).

Also, the interaction between the traveling material and the surrounding air affects significantly the system dynamic behavior. This is shown in several previously published works. Interaction of an axially moving band and surrounding fluid by boundary layer theory was studied by Frondelius et al. [16]. Modeling the dynamical behavior of a traveling paper web Kulachenko et al. [27,28] took into account the effects of the air. Recently, static and dynamic analysis of an axially moving plate interacting with axially moving ideal fluid were presented by Banichuk et al. [4,5].

The development of various kinds of nano-scale technologies will need to undertake a study into axially moving micro- and nano-objects. On the other hand the size dependence of deformation behavior in micro-scale had been experimentally observed in metals and polymers. Recently, miniaturized beams and plates have been widely used in micro- and nano-scale devices, modulators, resonators, and systems, such as biosensors and atomic force microscopes. All these miniaturized systems are expected to have a great influence on semiconductor technology, information technology and biology. The dynamic behavior of these systems cannot be explained by the conventional macro-scale theories and a special micro-scale model should be used. To the authors' knowledge only one study on axially moving nano-system has been published yet. In 2010 the nonlocal stress theory was used by Lim et al. [33] to derive the dynamic model of an axially moving nanobeam subjected to axial tension.

References

[1] Archibald FR, Emslie AG. The vibrations of a string having a uniform motion along its length. J Appl Mech ASME 1958;25:347–8.
[2] Ariaratnam ST, Namachchivaya NS. Dynamic stability of pipes conveying pulsating fluid. J Sound Vib 1986;107(2):215–30.
[3] Banichuk N, Jeronen J, Neittaanmaki P, Tuovinen T. On the instability of an axially moving elastic plate. Int J Solids Struct 2010;47:91–9.
[4] Banichuk N, Jeronen J, Neittaanmaki P, Tuovinen T. Static instability analysis for traveling membranes and plates interacting with axially moving ideal fluid. J Fluids Struct 2010;26:274–91.
[5] Banichuk N, Jeronen J, Neittaanmaki P, Tuovinen T. Dynamic behaviour of an axially moving plate interacting undergoing small cylindrical deformation submerged in axially flowing ideal fluid. J Fluids Struct 2011;27:986–1005.
[6] Chen L-Q. Analysis and control of transverse vibrations of axially moving strings. Appl Mech Rev ASME 2005;58:91–116.
[7] Chen L-Q, Ding H. Steady-state transverse response in coupled planar vibration of axially moving viscoelastic beams. J Vib Acoust 2010;132(1):011009 (9 pages).
[8] Chen L-Q, Yang X-D, Cheng C-J. Dynamic stability of an axially moving viscoelastic beam. Eur J Mech A/Solids 2004;23:659–66.
[9] Chen L-Q, Chen H. Asymptotic analysis of nonlinear vibration of axially accelerating viscoelastic strings with the standard linear solid model. J Eng Math 2010;67:205–18.
[10] Chen L-Q, Tang Y–Q. Parametric stability of axially accelerating viscoelastic beams with the recognition of longitudinally varying tensions. J Vib Acoust 2012;134(1):011008 (11 pages).
[11] Chen L-Q, Tang Y–Q, Lim CW. Dynamic stability in parametric resonance of axially accelerating viscoelastic Timoshenko beams. J Sound Vib 2010;329:547–565.
[12] Chen L-Q, Zhang N-H, Zu JW. Bifurcation and chaos of an axially moving viscoelastic string. Mech Res Commun 2002;29:81–90.
[13] Chen L-Q, Zhang N-H, Zu JW. The regular and chaotic vibrations of an axially moving viscoelastic string based on forth order Galerkin truncation. J Sound Vib 2003;261:764–73.
[14] D'Elenterio GH, Huges PC. Dynamics of gyroelastic continua. J Appl Mech ASME 1984;51:415–22.
[15] Ding H, Chen L-Q, Yang S-P, Zhang G-C. Forced vibrations of supercritically transporting viscoelastic beams. J Vib Acoust 2012;134(5):051007 (11 pages).
[16] Frondelius T, Koivurova H, Pramila A. Interaction of an axially moving band and surrounding fluid by boundary layer theory. J Fluids Struct 2006;22:1047–1056.

[17] Fung R-F, Huang J-S, Chen. Y-C. The transient amplitude of the viscoelastic traveling string: an integral constitutive law. J Sound Vib 1997;201(2):153–67.
[18] Fung R-F, Huang J-S, Chen Y-C, Yao C-M. Non-linear dynamic analysis of the viscoelastic string with a harmonically varying transport speed. Comput Struct 1998;66:777–84.
[19] Ghayesh MH. Nonlinear transversal vibration and stability of an axially moving viscoelastic string supported by a partial viscoelastic guide. J Sound Vib 2008;314:757–74.
[20] Ghayesh MH, Amabeli M. Nonlinear dynamics of axially moving viscoelastic beams over the bucked state. Comput Struct 2012;112 and 113:406–21.
[21] Ghayesh MH, Amabeli M, Paddoussis MP. Nonlinear dynamics of axially moving plates. J Sound Vib 2013;332:391–406.
[22] Hatami S, Azhari M, Saadatpour MM. Free vibration of moving laminated composite plates. Compos Struct 2007;80:609–20.
[23] Hatami S, Ronagh HR, Azhari M. Exact free vibration analysis of axially moving viscoelastic plates. Compos Struct 2008;86:1738–46.
[24] Holmes PJ. Bifurcations to divergence and flutter in flow-induced oscillations: a finite dimensional analysis. J Sound Vib 1977;53:471–503.
[25] D.I.G. Jones, Handbook of viscoelastic vibration damping, John Willey & Sons Ltd 2001.
[26] Kim J, Cho J, Lee U, Park S. Modal spectral element formulation for axially moving plates subjected to in-plane axial tension. Comput Struct 2003;81: 2011–2020.
[27] Kulachenko A, Gradin P, Koivurowa H. Modeling the dynamical behaviour of a paper web. Part I. Comput Struct 2007;85:131–47.
[28] Kulachenko A, Gradin P, Koivurowa H. Modeling the dynamical behaviour of a paper web. Part II. Comput Struct 2007;85:148–57.
[29] Laukkanen J. FEM analysis of a traveling web. Comput Struct 2002;80 (24):1827–42.
[30] Lee HP, Ng TY. Dynamic stability of a moving rectangular plate subject to in-plane acceleration and force perturbations. Appl Acoust 1995;45:47–59.
[31] Lengoc L, McCallion H. Wide bandsaw blade under cutting conditions, part I: vibration of a plate moving in its plane while subjected to tangential edge loading. J Sound Vib 1995;186(1):125–42.
[32] Lengoc L, McCallion H. Wide bandsaw blade under cutting conditions, part II: stability of a plate moving in its plane while subjected to parametric excitation. J Sound Vib 1995;186(1):143–62.
[33] Lim CW, Li C, Yu J-L. Dynamic behaviour of axially moving nanobeams based on nonlocal elasticity approach. Acta Mech Sinica 2010;26:755–65.
[34] Lin CC. Stability and vibration characteristics of axially moving plates. Int J Solid Struct 1997;34(24):3179–90.
[35] Lin CC, Mote Jr CD. Equilibrium displacement and stress distribution in a two dimensional axially moving web under transverse loading. J Appl Mech ASME 1995;62:772–9.
[36] Luo ACJ, Hamidzadeh HR. Equilibrium and buckling stability for axially traveling plates. Comun Nonlinear Sci Nonlinear Simul 2004;9:343–60.
[37] Luo Z, Hutton SG. Formulation of a three-node traveling triangular plate element subjected to gyroscopic and in-plane forces. Comput Struct 2002;80:1935–44.
[38] Mahalingam S. Transverse vibrations of power transmission chains. Br J Appl Phys 1957;8:145–8.
[39] Martinez-Agirre. M, Elejabarrieta MJ. Characterisation and modelling of viscoelastically damped sandwich structures. Int J Mech Sci 2010;52:1225–33.
[40] Marynowski K. Non-linear vibration of axially moving orthotropic web. Mech Mech Eng 1999;4:131–6.
[41] Marynowski K. Two-dimensional rheological element in modeling of axially moving viscoelastic web. Eur J Mech A/Solids 2006;25:729–44.
[42] Marynowski K. Dynamics of the axially moving orthotropic web. Springer-Verlag; 2008.
[43] Marynowski K. Free vibration analysis of the axially moving Levy-type viscoelastic plate. Eur J Mech A/Solids 2010;29:879–86.
[44] Marynowski K. Dynamic analysis of an axially moving sandwich beam with viscoelastic core. Compos Struct 2012;94:2931–6.
[45] Marynowski K, Kapitaniak T. Kelvin–Voigt versus Burgers internal damping in modeling of axially moving viscoelastic web. Int J Non-Linear Mech 2002;37 (7):1147–61.
[46] Marynowski K, Kapitaniak T. Zener internal damping in modeling of axially moving viscoelastic beam with time-dependent tension. Int J Non-Linear Mech 2007;42(1):118–31.
[47] Meirovitch L. A new method of solution of the eigenvalue problem in gyroscopic systems. AIAA J 1974;12(10):1337–42.
[48] Mockersturm EM, Guo J. Nonlinear vibration of parametrically excited, viscoelastic, axially moving strings. J Appl Mech ASME 2005;72:374–80.
[49] Moon J, Wickert JA. Non-linear vibration of power transmission belts. J Sound Vib 1997;200(4):419–31.
[50] Mote Jr CD. A study of band saw vibrations. J Frankl Inst 1965;279(6):430–44.
[51] Mote Jr CD. On the non-linear oscillation of an axially moving string. J Appl Mech ASME 1966;33:463–4.
[52] Mote Jr CD. Divergence buckling of an edge-loaded axially moving band. Int J Mech Sci 1968;10(4):281–95.
[53] Mote Jr CD. Stability of systems transporting accelerating axially moving materials. J Dyn Syst Meas Control ASME 1975;97:96–8.
[54] Nagarkatti SP, Zhang F, Costic BT, Dawson DM, Rahn CD. Speed tracking and transverse vibration control of an axially accelerating web. Mech Syst Signal Process 2002;16:337–56.
[55] Öz H, Pakdemirli M. Vibrations of an axially moving beam with time-dependent velocity. J Sound Vib 1999;227:239–57.
[56] Öz H, Pakdemirli M, Boyaci H. Nonlinear vibration and stability of an axially moving beam with time-dependent velocity. Int J Non-Linear Mech 2001;36:107–15.
[57] Pakdemirli M, Ulsoy AG. Transverse vibration of an axially accelerating string. J Sound Vib 1994;169(2):179–96.
[58] Pakdemirli M, Öz HR. Infinite mode analysis and truncation to resonant modes of axially accelerated beam vibrations. J Sound Vib 2008;311:1052–74.
[59] Pelicano F, Fregolent A, Bertizzi A, Vestroni F. Primary parametric resonances of a power transmission belt: theoretical and experimental analysis. J Sound Vib 2001;224(4):669–84.
[60] Pelicano F, Vestroni F. Non-linear dynamics and bifurcations of an axially moving beam. J Vib Acoust ASME 2000;122:21–30.
[61] Poidousis MP. Fluid–structure interactions. Slender structure and axial flow. London: Elsevier Academic Press; 2004.
[62] Pritz T. Analysis of four-parameter fractional derivative model of real solid materials. J Sound Vib 1996;195(1):103–15.
[63] Sack RA. Transverse oscillations in travelling strings. Br J Appl Phys 1954;5:224–6.
[64] Shin C, Chung J, Kim W. Dynamic characteristics of the out-of-plane vibration for an axially moving membrane. J Sound Vib 2005;286:1019–31.
[65] Simpson A. Transverse modes and frequencies of beams translating between fixed end supports. J Eng Sci 1973;15(3):159–64.
[66] Sobotka Z. Theory of plasticity and limit design of plates. Elsevier; 1989.
[67] Tang Y-Q, Chen L-Q. Nonlinear free transverse vibrations of in-plane moving plates: without and with internal resonances. J Sound Vib 2011;330:110–26.
[68] Tang Y- Q, Chen L- Q. Parametric and internal resonances of in-plane accelerating viscoelastic plates. Acta Mech 2012;223:415–31.
[69] Ulsoy AG, Mote Jr CD. Vibrations of wide band saw blades. J Eng Ind ASME, 104; 1982; 71–8.
[70] Wang H, Logghe D, Miskin D. Physical modelling and control of lateral web position for wallpaper making processes. Control Eng Pract 2005;13:401–12.
[71] Wang X. Numerical analysis of moving orthotropic thin plates. Comput Struct 1999;70:467–86.
[72] Wickert JA. Non-linear vibration of a traveling tensioned beam. Int J Non-Linear Mech 1992;27(3):503–17.
[73] Wickert JA, Mote Jr CD. Classical vibration analysis of axially-moving continua. J Appl Mech ASME 1990;57:738–44.
[74] Wickert JA, Mote Jr CD. Current research on the vibration and stability of axially moving materials. Shock Vib Digest 1988;20:3–13.
[75] Yang X-D, Zhang W, Chen L-Q, Yao M-H. Dynamical analysis of an axially moving plate by finite difference method. Nonlinear Dyn 2012;67:997–1006.
[76] Yang X-D, Chen L-Q. Bifurcation and chaos of an axially accelerating viscoelastic beam. Chaos Solut Fractals 2005;23(1):249–58.
[77] Yang X-D, Chen L- Q, Zu JW. Vibrations and stability of an axially moving rectangular composite plate. J Appl Mech 2010;78(1):011018 (11 pages).
[78] Young GE, Reid KN. Lateral and longitudinal dynamic behaviour and control of moving webs. J Dyn Syst Meas Control ASME 1993;115:309–17.
[79] Young GE, Shelton JJ, Kardamilas C. Modeling and control of multiple web spans using state estimation. J Dyn Syst Meas Control ASME 1989;111:505–10.
[80] Zhang L, Zu JW. Nonlinear vibration of parametrically excited moving belts, part I: dynamic response. J Appl Mech ASME 1999;66:396–402.
[81] Zhang L, Zu JW. Nonlinear vibration of parametrically excited moving belts, part II: stability analysis. J Appl Mech ASME 1999;66:403–9.
[82] Zhou YF, Wang ZM. Transverse vibration characteristics of axially moving viscoelastic plate. Appl Math Mech 2007;28:209–18.
[83] Zhou YF, Wang ZM. Vibrations of axially moving viscoelastic plate with parabolically varying thickness. J Sound Vib 2008;316:198–210.