International Journal of Non-Linear Mechanics 70 (2015) 84–94

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Synchronous rotational motion of parametric pendulums

A. Najdecka [a,*], T. Kapitaniak [b], M. Wiercigroch [a]

[a] Centre for Applied Dynamics Research, School of Engineering, University of Aberdeen, UK
[b] Division of Dynamics, Technical University of Lodz, Poland

ARTICLE INFO

Article history:
Received 27 December 2013
Received in revised form
15 August 2014
Accepted 8 October 2014
Available online 18 October 2014

Keywords:
Parametric pendulum
Synchronization
Coupled oscillators
Rotations
Experimental studies

ABSTRACT

We study theoretically and experimentally the synchronization phenomenon of two rotating parametric pendulums attached to common elastic support under harmonic excitation. Two types of synchronous states have been identified – complete and phase synchronization. The interactions in the system have been investigated numerically and experimentally. The relation between the synchronization mode and the stability of the rotational motion for a system with flexible support has been studied. It has been demonstrated that the synchronization of pendulums rotating in antiphase is more beneficial from energy harvesting viewpoint than the synchronization in phase. Finally, an influence of the parameter mismatch between the pendulums on their synchronization has been examined.

1. Introduction

The objective of this paper is to study the synchronization of rotational motion in the system of two parametric pendulums subjected to common harmonic excitation. The study has been motivated by a possibility of applying such a system for energy harvesting, as the oscillatory motion can be converted into rotation of pendulums. Consequently, the energy can be harvested from the rotational motion, which is a strongly advantageous alternative to using the energy of oscillations. The challenge of the design of such a structure lies in balancing it properly to guarantee dynamic stability once the pendulum is in motion. Therefore, to compensate for the effect which a single rotating mass exerts on the support, the system consisting of two pendulums is being considered. To achieve the desired balance of forces the pendulums would be required to counter rotate in a synchronized manner. If their responses are synchronized in antiphase the structure remains stable. This section provides an overview of the basics of synchronization theory, reviews the main recent works on the dynamics of parametric pendulum and looks at synchronization in pendulum systems.

1.1. Synchronization theory

The term ‘synchronous’ originates in Greek and denotes something ‘sharing the same time’. The discovery of the synchronization phenomena is directly related to the dynamics of the pendulum. It has been first observed and described in the 17th century by a Dutch researcher, Huygens, on an example of pendulum clocks hanging on the same wall [13]. Recently his experiment has been repeated in Kapitaniak et al. [14]. Huygens observation revealed that the clocks were exactly synchronized, swinging in opposite directions. Even if any disturbance occurred, they were still returning to the synchronized state after some transient time. The reason for this behaviour has been identified in the coupling effect of the beam supporting the clocks, transmitting the vibrations.

Since then synchronization has been detected in various systems and described in many publications. Pikovsky et al. [21] and Blekhman et al. [4] give examples of this phenomenon in mechanical, electrical or biological systems. In the most general sense, occurrence of synchronization between two systems implies existence of some relationship between their responses, without specifying exactly the type of this relation, which can be of a complex nature. Therefore, sometimes synchronization is difficult to detect, as it cannot always be associated with the identity of trajectories. Depending on the relation between the responses, several types of synchronization have been classified. Considering two systems, where $x(t)$ and $y(t)$ denote their trajectories, the following types of synchronization can be distinguished [5]: *Complete synchronization* [CS] is a state at which both phases and amplitudes of the oscillating systems coincide. It can be achieved only in case of identical oscillators when some kind of internal or external coupling between them is introduced. The definition of the CS concept has been introduced by Pecora and Carroll [19] and is said to be a state in which phase trajectories $x(t)$ and $y(t)$ of the coupled systems converge to the same value and

* Corresponding author.
E-mail addresses: anna.najd@gmail.com (A. Najdecka), tomaszka@p.lodz.pl (T. Kapitaniak), m.wiercigroch@abdn.ac.uk (M. Wiercigroch).

http://dx.doi.org/10.1016/j.ijnonlinmec.2014.10.008

remain in this relation during the further time evolution. The above concept can be described by the following relation:

$$\lim_{t\to\infty}|x(t)-y(t)|=0, \tag{1}$$

In practice very often the identity conditions are not fully met. If there is a difference in parameters or noise is present the *imperfect complete synchronization* [ICS] occurs and the synchronizability condition becomes

$$\lim_{t\to\infty}|x(t)-y(t)|<\epsilon, \tag{2}$$

where ϵ is a small parameter. *Phase synchronization* [PS] describes a weaker degree of synchronization. The required coupling between the systems is much lower than in case of CS so that the identity condition is not necessary. It occurs when the phases of oscillations are locked within a certain range. Generally speaking, this correlation does not imply any relation between the amplitudes. The mathematical condition for PS is given by

$$|n\Phi_1(t)-m\Phi_2(t)|<c, \tag{3}$$

where Φ_1 and Φ_2 denote phases of the coupled oscillators *n, m* integers determining the locking ratio and *c* is a constant. As a consequence, the frequencies of both systems ω_1 and ω_2 need to be locked as well and satisfy the relation

$$n\omega_1-m\omega_2=0, \tag{4}$$

Based on the type of the system in which synchronization is observed, another classification can be introduced. The first case, based on classical understanding of synchronization, is the synchronization of coupled periodic oscillators. The rhythms of self-sustained periodic oscillators adjust due to their weak interaction, where this adjustment can be described in terms of phase locking and frequency entrainment. The basic model of such coupled system consisting of two oscillators is given by

$$\frac{dx_1}{dt}=f_1(x_1)+\epsilon p_1(x_1,x_2),$$
$$\frac{dx_2}{dt}=f_2(x_2)+\epsilon p_2(x_2,x_1), \tag{5}$$

where ϵ is the coupling parameter. If ϵ vanishes the subsystems become independent and oscillate with their natural frequency. The second type of interaction considered here is the synchronization of periodic oscillators by external force. It can be also observed when a periodic force (or noise) is applied to a group of non-coupled autonomous oscillators. Its occurrence depends not only on the magnitude of forcing but also on the difference between the natural frequency of the system and the forcing one, called detuning parameter. Inside the synchronization region, the system oscillates with the frequency of the external force, while outside quasiperiodic motion can be observed.

Synchronization can also be observed in a noisy system. For such a system the condition for synchronization needs to be modified, for a less rigorous one. The perfect frequency entrainment is not observed any more. A state where frequencies nearly adjust, but still phase slips can be observed, is defined as imperfect phase synchronization (IPS). Finally, synchronization can be observed also for chaotic systems [25,15,6,20,26,23]. Its detection however depends on the type of attractor and can be more complex.

1.2. Parametric pendulum

The parametric pendulum is a system which has been of great interest for years, because of its rich dynamical behaviour [7,3,28,9]. It is a model with numerous engineering applications, including marine structures, superconductor Josephson junction. Many oscillating systems contain pendulum like non-linearity. Therefore, parametric pendulum has been one of the most common systems in the literature illustrating the dynamics of a non-linear oscillator. Among its various responses equilibrium points, oscillations, rotations as well as chaos can be observed.

The physical model of a parametric pendulum and the phase plane representation of the basic responses for unforced undamped system are shown in Fig. 1. The vertical oscillation of the pivot point results in the oscillations or rotations of the pendulum, depending on the initial conditions and forcing parameters. The closed loops marked by 1 and 2 correspond to the oscillations around hanging down position. Once the sufficient amount of energy is supplied the pendulum can escape from the potential well passing the critical case described by separatix (curve 3) and enter rotational motion regime (curves 4).

For many engineering applications, oscillatory responses are of main interest. Rotation of pendulum like systems has been studied before in relation to rotor dynamics and in recent years the research intensified due to potential applications in energy harvesting.

Approximating the escape zone has been the topic of study for Trueba et al. [32], Thompson [31], Bishop and Clifford [9] who used symbolic dynamics approach in their work. Different types of rotations have been classified in [8]. Xu and Wiercigroch [34] derived an analytical solution for rotational motion using multiple scales method where Sofroniou and Bishop [28] applied the harmonic balance method to the problem. Limit of rotational motion existence has been determined analytically by Koch and Leven [16] and Lenci et al. [17], who gave analytical approximation of the rotational solutions including study of their stability.

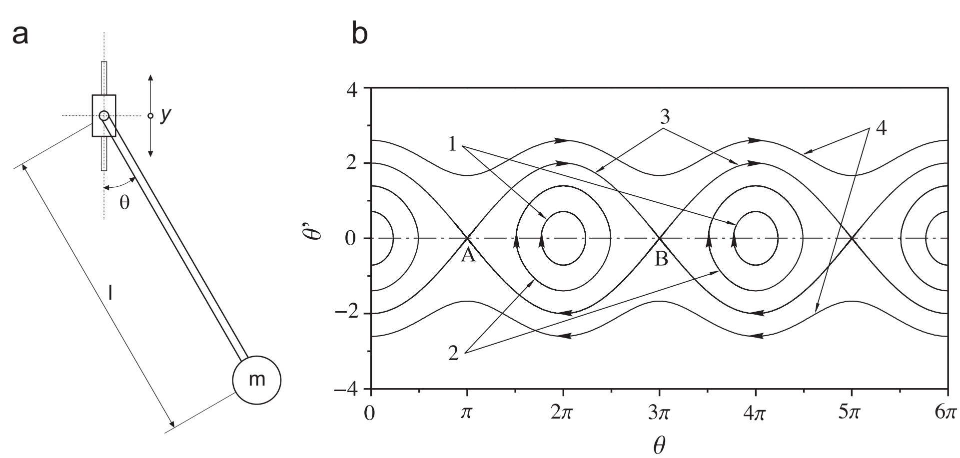

Fig. 1. (a) Physical model of parametrically excited pendulum and (b) phase plane showing different responses of the unperturbed pendulum in terms of pendulum displacement and velocity [35].

1.3. Synchronization in pendulums systems

Synchronous motion of pendulums is a classic example of synchronization, which has been studied in the literature for many years. It has been initiated by Huygens, based on the observation of the pendulum clocks. Senator [27] has analysed the model of two pendulum clocks, suspended at the common horizontal beam driven by the escapement mechanism, which can approximate the behaviour observed by Huygens. The synchronization is achieved due to horizontal elasticity of the supporting beam, which can oscillate together with the pendulums.

Banning et al. [2] have studied the dynamics of a system of two coupled parametrically excited pendulums. The four types of responses which were covered by the study comprise downward equilibrium position, synchronized in-phase oscillation, synchronized anti-phase oscillation and a mixed motion of two pendulums, corresponding to the lack of synchronization. Teufel et al. [30] studied the synchronized oscillation of the coupled pendulums, which can be described by the Van-der-Pol equations. The pendulums are connected by a linear spring and exposed at the steady fluid flow. Existence and stability of the synchronized solutions are investigated for the case of strong and weak coupling. The transition to escape from the synchronized state for an undamped pendulums system has been studied by Quinn [24]. Since the concept of synchronization has been transferred to chaotic systems, synchronization of chaotic pendulums has been investigated. Fradkov and Andrievsky [12] studied the synchronization phenomena for the system of two coupled pendulums, where one of them was forced by an external torque and coupled with the second one by a torsional spring. The external forcing plays a role of a control action.

The model of two coupled periodically forced chaotic pendulums was also studied by Baker et al. [1], where the coupling was unidirectional, so that the relation between the pendulums was of a 'master–slave' type. Zhang et al. [36] considered a case of synchronizing uncoupled parametric pendulums in their chaotic regime. Application of the control method, which would bring the chaotic trajectories starting from different initial conditions to coincide, was discussed. Synchronization control based on the pendulums energy level was proposed by Pogromsky et al. [22]. Most of the studies deal with synchronized oscillations or chaotic motion. In recent years along with more interest in rotational motion, the synchronized rotation of pendulums was studied. Synchronization in the system of rotating pendulums on the common movable support was studied recently in [11,10]. Different synchronization types were identified in the system with separate forcing applied directly to each pendulum. More recently an experimental study of four rotating double pendulums under common parametric excitation has been conducted by Strzalko et al. [29].

This paper is structured as follows. In Section 2 mathematical model of the two pendulums system is constructed. It is followed by a numerical study of the synchronization phenomena.

Results showing different synchronized responses of the system are presented, with the focus on the rotational motion. The in-phase and antiphase synchronization cases are compared from the point of view of stability of the rotational motion and the base response. Finally, the influence of the parameter mismatch on the response of the system and its synchronization is studied. In Section 4, the experimental setup and results are presented and compared with the numerical ones.

2. Modelling of rotating parametric pendulums system

The system considered in this study consists of the two pendulums mounted on the commonly excited flexible supporting structure. Initially, it has been modelled on a plane as a four-degrees-of-freedom system illustrated in Fig. 2(a), where x and y denote the displacement of the structure in the horizontal and the vertical direction respectively, θ_1 and θ_2 describe the angular displacements of the pendulums from the downward zero position, l_1 and l_2 denote their lengths. m and M are the masses of pendulum bobs (treated as a point masses) and supporting structure respectively. k_x, k_y, c_x, c_y represent the overall stiffness and damping properties of the pendulum support in the horizontal and the vertical direction. Synchronized state can be achieved due to coupling effect of the elastic base, capable of transmitting vibrations between the pendulums, as well as the common harmonic forcing applied at the base in the vertical direction. The equations of motion for the system have been derived using the Lagrange energy method [18]:

$$
\begin{aligned}
&(M+2m)\ddot{X}+c_x\dot{X}+k_xX++ml_1\left(\ddot{\theta}_1\cos\theta_1-\dot{\theta}_1^2\sin\theta_1\right)\\
&\quad+ml_2\left(\ddot{\theta}_2\cos\theta_2-\dot{\theta}_2^2\sin\theta_2\right)=0,\\
&(M+2m)\ddot{Y}+c_y\left(\dot{Y}-\dot{R}_y\right)\\
&\quad+k_y(Y-R_y)++ml_1\left(\ddot{\theta}_1\sin\theta_1+\dot{\theta}_1^2\cos\theta_1\right)\\
&\quad+ml_2\left(\ddot{\theta}_2\sin\theta_2+\dot{\theta}_2^2\cos\theta_2\right)=0,\\
&ml_1^2\ddot{\theta}_1+ml_1\\
&\quad\times\left(\ddot{X}\cos\theta_1+\ddot{Y}\sin\theta_1+g\sin\theta_1\right)+c_\theta l_1\dot{\theta}_1=0,\\
&ml_2^2\ddot{\theta}_2+ml_2\\
&\quad\times\left(\ddot{X}\cos\theta_2+\ddot{Y}\sin\theta_2+g\sin\theta_2\right)+c_\theta l_2\dot{\theta}_2=0,
\end{aligned}
\tag{6}
$$

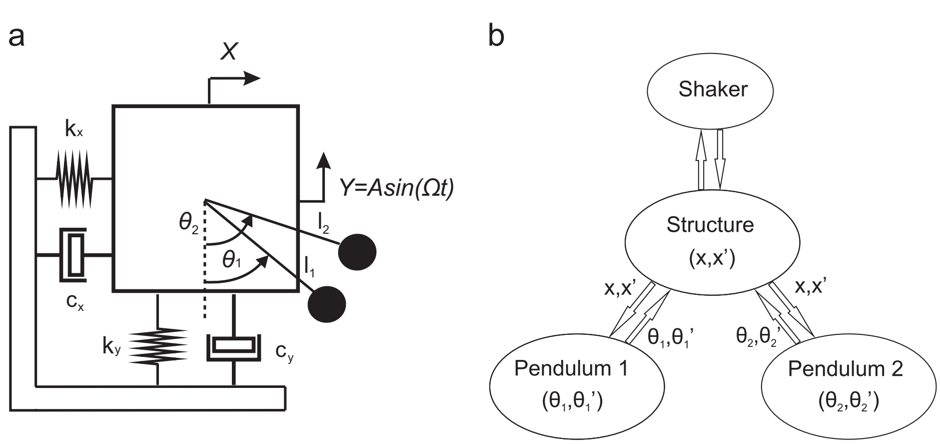

Fig. 2. (a) Physical model of the pendulum system and (b) schematic representation of the interactions between the synchronized subsystems.

where $R_y = A \sin(\Omega t)$ represents the harmonic excitation applied to the base. In the first stage of the study pendulums are assumed to be identical ($l = l_1 = l_2$). The preliminary studies of the dynamics of the system with parameters corresponding to the experimental set-up revealed that due to the high stiffness of the base in the vertical direction k_y the degree-of-freedom describing the vertical displacement of the structure Y can be neglected. Hence the displacement of the pendulum pivot in vertical direction is equal to the displacement of the pendulum base, $Y = R_y = A \sin(\Omega t)$. Then the system can be simplified to three degrees-of-freedom and Eq. (6) can be rewritten as

$$\begin{aligned}
&(M+2m)\ddot{X} + c_x\dot{X} + k_xX + +ml \\
&\times\left(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 \sin\theta_1 + \ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2\right) = 0, \\
&ml^2\ddot{\theta}_1 + ml\left(\ddot{X} \cos\theta_1 - \Omega^2 A \sin\Omega t \sin\theta_1 + g \sin\theta_1\right) + c_\theta l\dot{\theta}_1 = 0, \\
&ml^2\ddot{\theta}_2 + ml\left(\ddot{X} \cos\theta_2 - \Omega^2 A \sin\Omega t \sin\theta_2 + g \sin\theta_2\right) + c_\theta l\dot{\theta}_2 = 0.
\end{aligned} \tag{7}$$

After non-dimensionalization with respect to the natural frequency $\omega_n = \omega_{n1} = \omega_{n2}$, so that $\tau = t\omega_n$ the equations of motion for the two pendulums can be rewritten in terms of new parameters:

$$\begin{aligned}
&\gamma_x = \frac{c_x}{(M+2m)\omega_n}, \quad \gamma_\theta = \frac{c_\theta}{m\omega_n}, \quad \alpha_x = \frac{k_x}{(M+2m)\omega_n^2}, \\
&a = \frac{m}{M+2m}, \quad p = \frac{A}{l}, \quad \omega = \frac{\Omega}{\omega_n}.
\end{aligned} \tag{8}$$

Substituting (8) into (7)

$$\begin{aligned}
&x'' + \gamma_x x' + \alpha_x x + \\
&\quad + a\left(\theta_1'' \cos\theta_1 - \theta_1'2 \sin\theta_1 + \theta_2'' \cos\theta_2 - \theta_2'2 \sin\theta_2\right) = 0, \\
&\theta_1'' + x'' \cos\theta_1 + (1 - \omega^2 p \sin(\omega\tau)) \sin\theta_1 + \gamma_\theta\theta_1' = 0, \\
&\theta_2'' + x'' \cos\theta_2 + (1 - \omega^2 p \sin(\omega\tau)) \sin\theta_2 + \gamma_\theta\theta_2' = 0,
\end{aligned} \tag{9}$$

where all of the system parameters are non-dimensional: γ_x is the damping coefficient of the base in the horizontal direction, α_x is the stiffness coefficient, p is the forcing amplitude, ω is the forcing frequency, τ is the time, a is a mass ratio. The last two equations describe the motion of the two pendulums. There is no explicit coupling term included. Instead the coupling effect occurs through the vibration of the common support in the horizontal direction (x'') and the parametric forcing applied in the vertical direction ($p \sin(\omega\tau)$). The mutual interaction of the subsystems is represented schematically in Fig. 2(b).

There are two types of interactions in this system. Firstly, the two pendulums tend to synchronize thanks to their identity and common excitation. Secondly, the two pendulum subsystems are not totally independent due to coupling effect of the elastic base. The coupling coefficient is not given explicitly but the coupling strength depends on the parameters of the system and their mutual relation including mass ratio of single pendulum to the total mass of the structure, stiffness and damping of the support. If X_1, X_2, X_3 denote the state variables of the first and second pendulums and of the oscillating support, then the equations of motion of the coupled system are given by

$$\begin{aligned}
&\ddot{X}_1 = f(X_1) + v(X_1)G(X_1, X_2) + q(X_1)F(t), \\
&\ddot{X}_2 = f(X_2) + v(X_2)G(X_1, X_2) + q(X_1)F(t), \\
&G(X_1, X_2) = \ddot{X}_3 = h(X_3) + g(X_1, X_2),
\end{aligned} \tag{10}$$

where f describes the dynamics of a single unforced pendulum, $F(t)$ is a harmonic function corresponding to the vertical forcing applied at the base. $G(X_1, X_2)$ represents the total coupling function and corresponds to the horizontal acceleration of the base, v and q are the trigonometric functions, linking G and F to $\ddot{X}_1, \ddot{X}_2$. Function h governs the dynamics of the base and function g includes coupling terms. The total coupling strength depends on the properties of the pendulum base and can be described by a set of coefficients: $[\gamma_x, \alpha_x, a]$.

3. Numerical study and analysis

In this section different types of synchronization described theoretically in the introduction have been observed numerically in the pendulum system. The relationship between the synchronized states and base response is shown. The second subsection demonstrates the influence of the synchronization type on the rotational motion of the pendulums. Finally, the case of non-identical pendulums is considered and the detected synchronized states are discussed.

3.1. Types of synchronization

To understand the overall dynamics of the system described above, a set of three second order differential equations needs to be solved. For a single pendulum a solution can be obtained using perturbation methods [35,34,17]. In this case due to the indirect coupling between the equations, which does not allow for uncoupling and solving them separately, analytical solutions lead to very complex expressions. Therefore, to study the dynamics and interaction between the two pendulums and their support the response of the system has been simulated numerically in terms of 6 basic state variables. The set of parameter values corresponding to the experimental set-up has been used (Table 1).

After non-dimensionalization it corresponds to Table 2.

To detect the occurrence of synchronization in the system new variables z and z^* have been introduced, defined as

$$z = \theta_1 + \theta_2, \quad z^* = \theta_1 - \theta_2 \tag{11}$$

For complete synchronization in antiphase $z = 0$ is required [19]. Complete synchronization in antiphase can be detected when $z^* = 0$. Numerical simulations of the system response under harmonic excitation revealed that the pendulums exhibit natural tendency to synchronize, irrespective of initial conditions and particular attractor. The correlation of phases between the two pendulums and base motion is visible in Figs. 3–5 showing the time histories of phase variables of the system x, θ_1, θ_2 for different forcing parameters resulting in various synchronized responses. These include synchronized in antiphase period one rotation of both pendulums (Fig. 3), synchronized in phase rotation (Fig. 4), rotation of one pendulum synchronized with oscillation of the second one (Fig. 5). Two types of synchronized responses have been displayed in Fig. 6. The numerical phase portraits of the synchronized pendulums are compared with the

Table 1
System parameters.

m (kg)	l (m)	M (kg)	k_x (N/m)	c_x (kg/s)	c_θ (kg/s)
0.709	0.271	11.700	5.61 × 105	622	0.0520

Table 2
Non-dimensional system parameters.

a	α_x	γ_x	γ_θ
0.05405	1181.3960	7.8808	0.01219

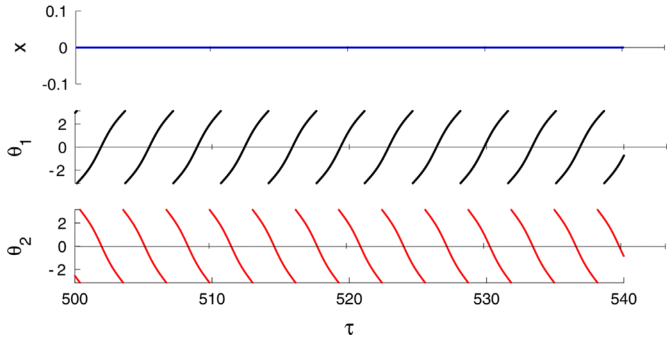

Fig. 3. Numerical time evolution of horizontal vibration of the base and angular displacements θ_1, θ_2 for $p=0.07$, $\omega=2$, showing complete synchronization of period one rotations in antiphase between the two pendulums.

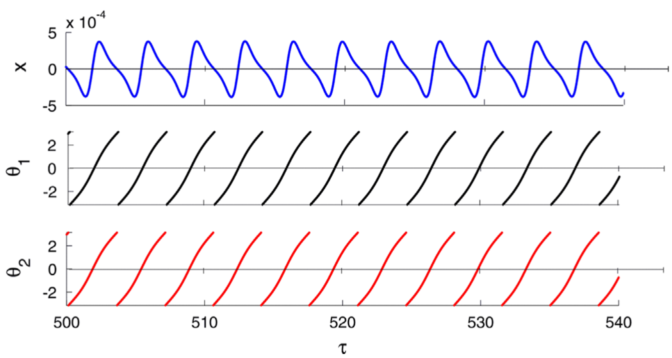

Fig. 4. Numerical time history of horizontal vibration of the base and angular displacements θ_1, θ_2 for $p=0.07$, $\omega=1.8$, showing complete synchronization in antiphase of the period one rotation of both pendulums; phase synchronization with the vibrating base x.

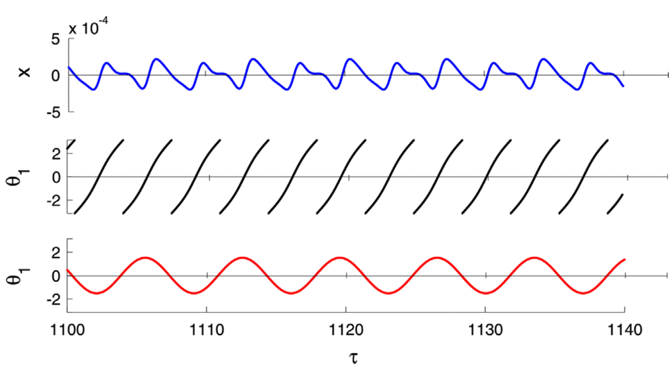

Fig. 5. Numerical time evolution of horizontal vibration of the base and angular displacements θ_1, θ_2 for $p=0.07$, $\omega=1.8$, showing phase synchronization of period one rotation of pendulum 1 with oscillation of pendulum 2 (locking ratio 2:1).

experimental results and a very good correspondence can be observed. If both pendulums start from the initial conditions within the same rotational attractor the resultant motion is a synchronized in phase rotation whereas for the opposite attractors synchronization in antiphase can be observed. Similar behaviour can be observed for oscillations. When the trajectories of rotational or oscillatory motion coincide, the observed phenomenon is the complete synchronization. If the two pendulums start within different basins of attraction then the synchronization between the oscillations and rotations can be observed. For such a case the frequencies and phases of the two pendulums lock, while their amplitudes remain independent, i.e. the phase synchronization occur. Considering synchronization of rotational motion of pendulums under arbitrary forcing conditions different arrangements of the phase shift can occur.

In this case due to the periodic parametric forcing, only two modes of synchronization between them are possible. To maintain stable rotational response pendulums need to be in phase with the external excitation, so that the movement of the base results in energy input to the pendulum and consequently maintains the rotational motion. Therefore both pendulums are required to rotate in phase with each other (with zero phase shift) or in antiphase (with a phase shift approaching π), to be in phase with the excitation. For complete synchronization in phase,

$$\theta_1=\theta_2, \quad \dot\theta_1=\dot\theta_2, \quad \ddot\theta_1=\ddot\theta_2, \tag{12}$$

while for the synchronization in antiphase,

$$\theta_1=-\theta_2, \quad \dot\theta_1=-\dot\theta_2, \quad \ddot\theta_1=-\ddot\theta_2. \tag{13}$$

For the synchronized in phase case, the coupling function becomes

$$g(\theta_1,\dot\theta_1,\ddot\theta_1,\theta_1,\dot\theta_1,\ddot\theta_1)=2a\left(\theta_1'' \cos\theta_1-\theta_1'^2 \sin\theta_1\right). \tag{14}$$

In contrast, for the synchronized in antiphase case, the coupling function vanishes:

$$g(\theta_1,\dot\theta_1,\ddot\theta_1,\theta_1,\dot\theta_1,\ddot\theta_1)=0. \tag{15}$$

As a result, when substituting Eq. (13) into system equation (15) it can been seen that

$$x=\dot x=\ddot x=0. \tag{16}$$

Consequently, there are no vibrations of the structure in the horizontal plane. Even though a single rotating pendulum would excite the base to oscillate in the horizontal direction (Fig. 5), this effect can be avoided if the two pendulums rotate in antiphase. This property of the synchronized state to damp the lateral vibrations has been confirmed in numerical simulations (Fig. 3). Fig. 7 shows the phase planes of the horizontal oscillations for the two pendulums rotating in antiphase as opposed to the rotation in phase or rotation of a single pendulum while the other one oscillates. The vibrations initially induced in the structure are represented by a black line and correspond to the transient response. As the pendulums reach the steady state, so does the oscillation of the support. It settles down on a periodic attractor (for pendulums in phase or a single pendulum rotating) or goes to equilibrium at zero (for synchronized in antiphase case), denoted by red colour. For the in phase synchronization the horizontal response of the base is similar to the single pendulum, however the amplitude of oscillation is magnified, due to the effects of both pendulums summing up. If one of the pendulums oscillates and the other one rotates, the response will be a period two motion due to the periodicity of the oscillation.

3.2. Influence of the synchronization type on the rotational motion of the pendulums

Synchronization of the two pendulums occurs due to the work done by the synchronizing torque with which the support acts on each pendulum [11,10]. In this case synchronizing torque consists of an external forcing part and the component from the vibration induced in the base and is given by

$$T_S^i=ml\left(\ddot X \cos\theta_i+\ddot Y \sin\theta_i\right)=ml\left(\ddot X \cos\theta_i-\Omega^2 A \sin\Omega t \sin\theta_i\right), \tag{17}$$

which in terms of non-dimensional parameters becomes

$$t_S^i=\ddot x \cos\theta_i+\ddot y \sin\theta_i=\ddot x \cos\theta_i-p \sin\omega\tau \sin\theta_i, \tag{18}$$

where the first part of the expression corresponds to the coupling effect of the base and contributes to synchronization between pendulums, while the second part represents the effect of the common excitation and results in synchronization with the harmonically excited base.

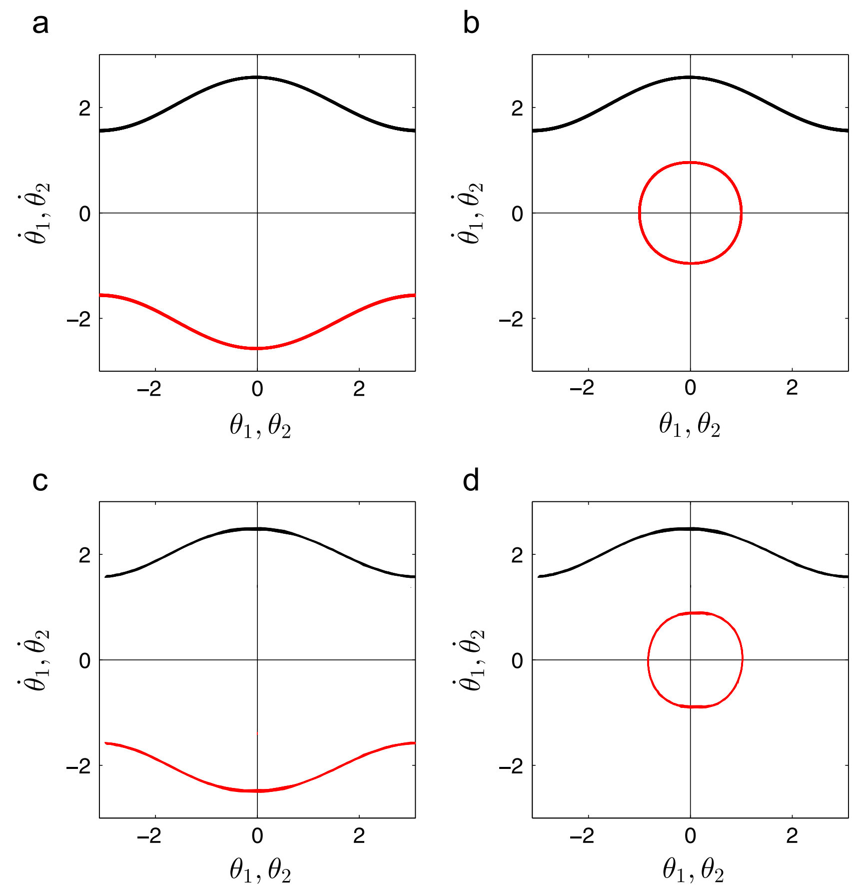

Fig. 6. Numerical results for $\omega=2$, $p=0.07$ showing (a) synchronized in anti-phase period one rotations, (b) rotation synchronized with oscillation for $\omega=2$, $p=0.07$ and (c, d) experimental verification, $f=2$ Hz, $A=0.012$ m.

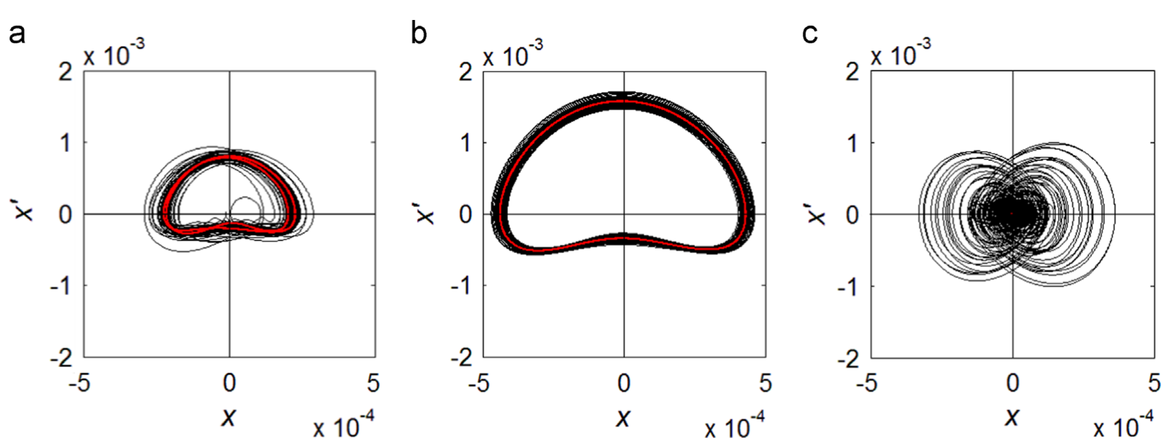

Fig. 7. Horizontal vibrations of the structure induced by different responses of the pendulum system: (a) one of the pendulums rotating while the other one oscillates; (b) synchronized in phase rotation of two pendulums; (c) synchronized in antiphase rotation of two pendulums. Black line denotes the transient response of the base and red the steady state response: (a) period two oscillation, (b) period one oscillation and (c) equilibrium at 0. (For interpretation of the references to colour in this figure caption, the reader is referred to the web version of this paper.)

From Eq. (16) for the two pendulums rotating in opposite directions, i.e. synchronized in antiphase $\ddot{x}=0$. Therefore, there is no energy transfer between the two pendulums.

As a consequence all the energy input by the excitation is being transferred by the vertical movement of the base to the rotating pendulums, while no energy is being used on exciting the base in the horizontal direction. Comparing the in phase and in antiphase synchronization, the latter one represents more beneficial state for many applications as it maintains the system at the lower energy level. Consequently, the minimum amount of energy which is required to maintain a stable rotation of the pendulums in phase with each other is higher than the one for antiphase motion. To confirm that lower limit of existence of rotational motion in forcing parameters phase space has been studied for different cases.

Fig. 8(a) shows the numerically computed saddle-node (SN) bifurcation curve in the parameter space (ω, p), corresponding to the lower limit of existence of rotational solutions for the system considered. Three cases are compared, a single rotating pendulum

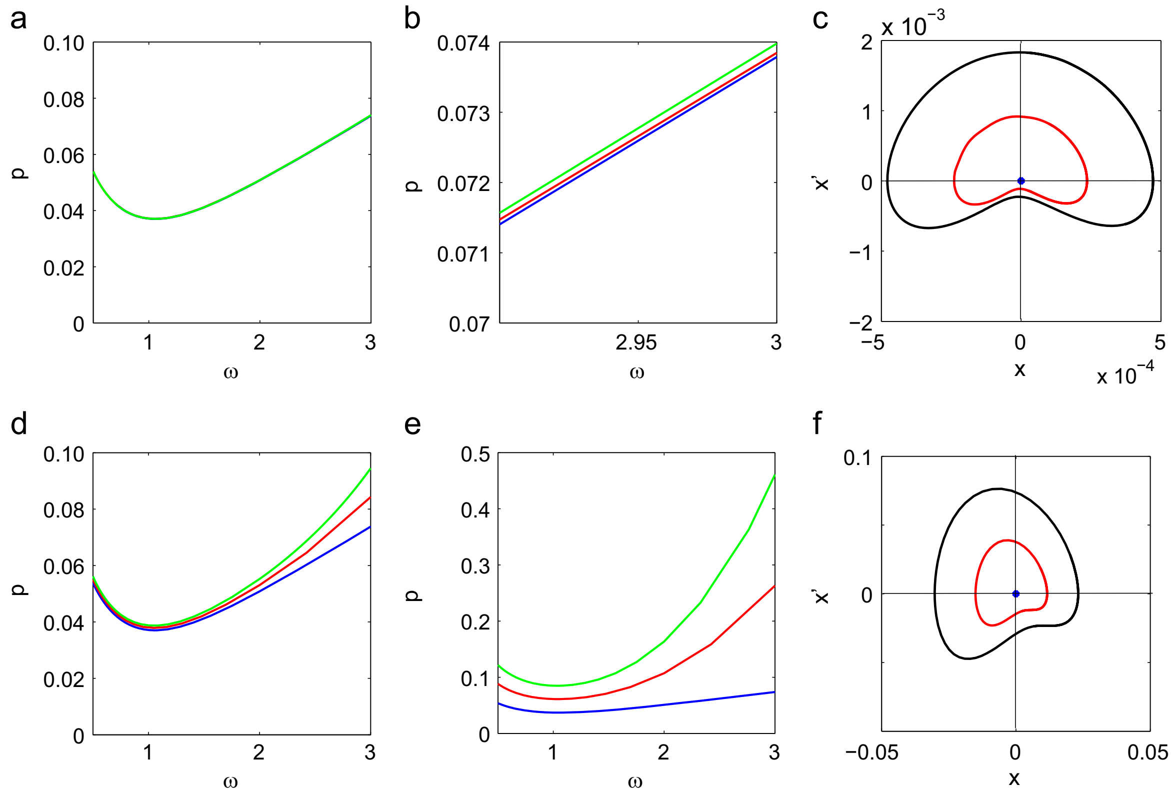

Fig. 8. Left: numerically computed in MATCONT, saddle node bifurcation curves corresponding to lower boundaries of rotation for two pendulums counter rotating (blue), rotating in the same direction (green) and a single rotating pendulum (red) and varying base stiffness. (a) Original value of α_x, (b) zoomed area of (a), (d) for $\alpha_x/10$, (e) for $\alpha_x/100$. Right: phase planes comparing the oscillations of the base in the horizontal direction for ω=2, p=0.2 and a different stiffness, (c) α_x, (f) $\alpha_x/100$; two pendulums rotating in phase (black), rotating in antiphase (blue), single pendulum rotating (red). (For interpretation of the references to colour in this figure caption, the reader is referred to the web version of this paper.)

(i.e. the other one has been fixed), two pendulums rotating in the same direction and two pendulums rotating in opposite directions. At the right side of the figure the curves seem to overlap, however from Fig. 8(b) containing zoomed plot area the difference between the curves can be seen. It is visible that introducing the two pendulums rotating in antiphase shifts the lower limit of rotational motion down the p-axis with respect to the case of one independently rotating pendulum. If the pendulums synchronize in phase with each other, then the limit curve shifts upwards. For a very stiff system like the one corresponding to the experimental rig (Table 2), this difference is not significant. To investigate what would be the effect of the synchronization mode on existence of rotational motion, if structural changes were introduced, two further cases were considered in the numerical study.

Firstly, the stiffness of the base has been decreased by factor 10 and the computations of the lower boundary of rotation were repeated. In Fig. 8(d) three separate characteristics appearing in the same order as in Fig. 8(b) can be clearly distinguished. Similarly Fig. 8(e) shows the boundary curves for stiffness parameter decreased by factor 100. There is a significant difference between them, which can be observed in the whole frequency range. The rotational solution synchronized in antiphase starts existing for much lower forcing amplitudes compared to a single rotating pendulum, while for the synchronized in phase pendulums much higher forcing amplitudes are required.

To understand the change in the pendulum dynamics caused by the decreased lateral stiffness of the pendulum support the horizontal vibrations in the base have been studied. For the initial stiff system the horizontal response of the base is three orders smaller than the vertical excitation supplied to the system (Fig. 8(c)). For the case of two pendulums rotating in antiphase the base

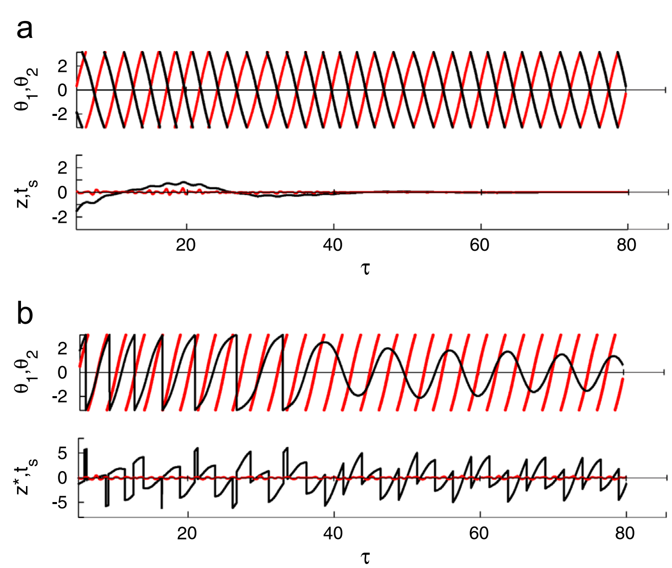

Fig. 9. Evolution of the pendulum displacement (top), phase difference and synchronizing torque (bottom) for pendulums with different initial conditions for ω=2.5, p=0.2, $\alpha_x/100$. (a) Synchronization in antiphase achieved for counter rotating pendulums, with initial conditions $\dot{\theta}_{10}=2.5, \dot{\theta}_{10}=-2.5$ and $z_0=-1$. (b) Lack of synchronization for $\dot{\theta}_{10}=2.5, \dot{\theta}_{10}=2.5$, $z_0^*=1$.

remains horizontally still. A single pendulum rotating while keeping the other one fixed excites the base to oscillate. If two pendulums rotate in phase, their effect on the base sums up and as a result the amplitude of the horizontal oscillation of the support increases. Fig. 8(c, f) shows how the amplitude of the base oscillations increases with the decreasing stiffness of the support.

Reducing the stiffness by factor 100 results in the amplitude of base oscillation increasing 100 times.

Apart from the base stiffness, the dynamic responses of the support depend on rotational speed of the pendulum $\dot{\theta}$ and indirectly on the forcing frequency ω. For rotational motion of both pendulums at higher frequencies, the resultant force acting on the base increases, which results in higher amplitude of the horizontal base oscillations and faster energy transfer between the pendulums. Therefore, the difference between the antiphase and the in-phase limit of rotational motion is mostly visible for high ω. To see how this fact affects the evolution of two rotational solutions, initially shifted in phase, a set of numerical simulations has been performed on pendulums rotating in the same and the opposite direction. In the first case the pendulums have been given the following initial conditions, $\theta_{10} = 1.5$, $\theta_{20} = -2.5$, i.e. $z_0 = -1$, and opposite angular velocities $\dot{\theta}_{10} = 2.5$, $\dot{\theta}_{10} = -2.5$, to ensure counter rotation. Fig. 9(a) shows the resultant time histories of the angular displacements and phase variable z. After the transient time the initial phase shift disappears, z goes to zero and pendulums synchronize completely in antiphase.

The evolution of the total synchronization torque shows how from initially high value it decays as the synchronized state is achieved. Fig. 9(b) depicts the results for two pendulums rotating initially in the same direction with the same phase shift as in the first case $z^*_0 = 1$ and $\theta_{10} = 1.5$, $\theta_{20} = 2.5$, and with the same rotational speed $\dot{\theta}_{10} = 2.5$, $\dot{\theta}_{10} = -2.5$. In this case the synchronized state is not achieved and one of the pendulums loses the rotational motion. This difference between the behaviour the two pendulums rotating in the opposite and in the same direction confirms that the synchronization in antiphase is a preferred configuration for the system, as it requires less energy to be maintained and can be initiated easier than the synchronization in phase.

3.3. Phase synchronization of non-identical pendulums

In practical application we often have to do with imperfections of mechanical parts. Hence this subsection deals with a case of non-identical pendulums, which results in a less obvious type of synchronization. To study the synchronization of non-identical pendulums the difference in their lengths has been introduced, which results in different natural frequencies. The mismatch of the parameters is given by

$$\delta = l_1 - l_2, \tag{19}$$

or as a non-dimensional number with regard to the original length,

$$\Delta = \frac{l_1 - l_2}{l_1}. \tag{20}$$

The results from the numerical studies of counter rotating pendulums showing the time histories of the phase sum z for different values of parameter mismatch have been summarized in Fig. 11. The system does not achieve complete synchronization state any more as z is non-zero. Instead

$$\lim_{t\to\infty} z < \epsilon, \tag{21}$$

where ϵ is a small number. The phase sum initially oscillates with high amplitude and finally settles down to periodic oscillations as the transient elapses, with a small finite amplitude dependent on the parameter mismatch. The transient response together with the steady state oscillation of the phase sum corresponding to different values of length mismatch is also shown in Fig. 11.

Such type of synchronization has been defined by Kapitaniak et al. [15] as ‘practical synchronization’. Some works refer to it as ‘almost complete synchronization’ [14]. If the difference in lengths is increased further, z increases until the critical value $\Delta = 0.475$ when the second pendulum loses its rotational motion and the responses of the two pendulums become asynchronous. In the case of two pendulums rotating in the same direction, desynchronizing occurs earlier. For the stiff system considered here, the limit value is $\Delta = 0.47$. The change of the length of one of the pendulums results in different natural frequencies and in consequence the shifted resonance structure. Therefore, when increasing the forcing amplitude, the transition to chaos does not occur at the same time for

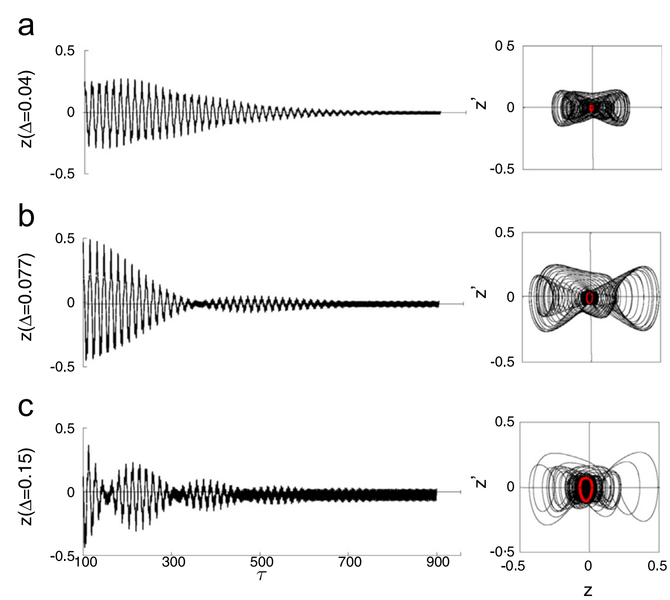

Fig. 11. Time histories of phase variable z and corresponding phase planes for different values of parameter mismatch, $\Delta = 0.04$, $\Delta = 0.077$, $\Delta = 0.15$, for $\omega = 2$, $p = 0.07$.

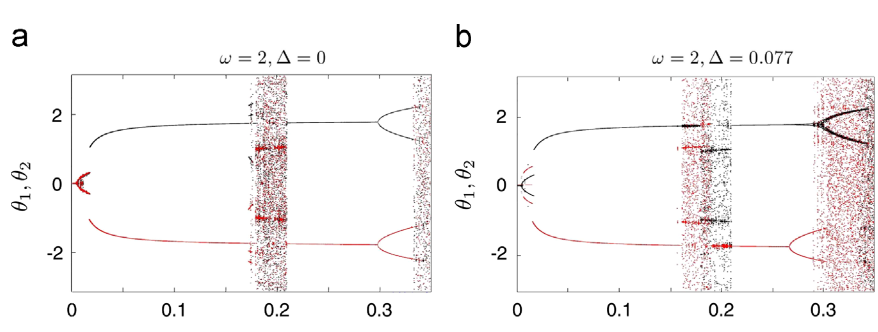

Fig. 10. Comparison of the bifurcation diagrams for identical and non-identical pendulums for $\omega = 2$ and changing amplitude of forcing, where black denotes response of pendulum 1 (fix length) and red pendulum 2 (whose length has been changed). (For interpretation of the references to colour in this figure caption, the reader is referred to the web version of this paper.)

both pendulums. The shorter pendulum escapes faster, causing an asynchronous region in the parameter space, when one of the pendulums is still in the periodic rotational mode while the other one behaves chaotically. The difference between identical and different pendulums is visible in their bifurcation diagrams (Fig. 10). For the $\Delta = 0$ case complete synchronization is maintained for all values of p, while for $\Delta = 0.077$ the pendulums are synchronized only in certain parameter ranges, where their basins of attraction overlap. Besides for the non-identical case complete synchronization cannot be achieved, instead in the synchronous regions the phase shift is always present, which is mostly visible for the oscillatory solutions in Fig. 10.

4. Experimental results

4.1. Experimental set-up

To verify the numerical results on different synchronization modes observed and study their correspondence to the real conditions, the set of experiment has been conducted on the rig corresponding to the parameters used in the numerical simulations. The studies have been carried out in the Centre of Applied Dynamics Research at the University of Aberdeen.

The experimental set-up used in this work consists of the system of two pendulums fixed on the common base and the electrodynamic shaker. The pendulum rig has been mounted on an electrodynamic shaker (Fig. 12). The oscillation of the shaker provides a parametric excitation to the system. The pendulum rig consists of two independent pendulums with bob masses at the ends, threaded to thin steel rods. The bob masses are brass disks with a threading inside. The threaded connection with the rod allows easy modification of the set-up by exchanging the connecting rod to change the length of the pendulum. Therefore, the system allows us to study the response of two identical as well as non-identical pendulums. The rods are fixed to two independent concentric shafts, supported by needle bearings at each side to minimize the friction. Each shaft has an encoder attached to it and a gear with a belt coupling the shaft to the servo-motor. The measurements from the encoders are input to the PC via NI board and processed in LabView (Fig. 12), where based on the inputs the control signal can be generated. The excitation to the shaker has been provided by a WaveTek generator. The interactions in such pendulum shaker system have been studied by Xu [33], who developed a complex model of the system including mechanical and electrical degrees-of-freedom.

The response of the pendulums has been observed for different initial conditions and changing forcing parameters. At first harmonic excitation with varying frequency and amplitude has been supplied to the shaker and the numerically obtained results have been verified experimentally. Then the system has been modified to demonstrate the change in the response, if the lengths of the pendulums vary.

4.2. Experimental results

When studying the response of the system under harmonic excitation no complete synchronization between the two pendulums can be observed. Due to the natural imperfections of the system the sum of phases is never constant and equal to zero; instead oscillates around a small value. The magnitude of the phase shift is different depending on the forcing frequency. For higher frequencies, the system approaches the complete synchronization (z close to 0) as shown in Fig. 13(i), while for the low frequencies the phase shift increases (Fig. 13(c)). In experimental conditions, the excitation provided to the structure is never perfectly harmonic.

Noise from the system is always present in the excitation of the shaker, however its intensity varies depending on the frequency. In general the shaker is more sensitive when working in the low frequency range below 1.5 Hz. Additionally, shaker as a dynamic system interacts with the rotating masses of the pendulums which causes perturbations to the harmonic signal, as discussed in [33]. As a consequence the experimentally observed synchronized state is an imperfect complete synchronization.

Finally, length of one of the pendulums has been reduced. Similar to the numerical study three different values of parameter mismatch have been considered: $\Delta = 0.04$, $\delta = 0.077$, $\delta = 0.15$. The results of the experiments with the two pendulums of different lengths confirmed numerical predictions. After the transient time the phase synchronization state has been observed, with the phase shift increasing with the length difference (Fig. 14). On the other hand, the influence of the length mismatch on the synchronization is smaller in the experimental conditions than for the numerical predictions. The responses of the system for different values of length mismatch Δ in the experimental conditions differ between each other less than in the numerical case, where pendulums were initially modelled to be identical.

5. Conclusions

The occurrence of the theoretically discussed stable synchronous states has been studied. The system considered consisted of two pendulums with a coupling common support. It has been demonstrated numerically and experimentally that even if the

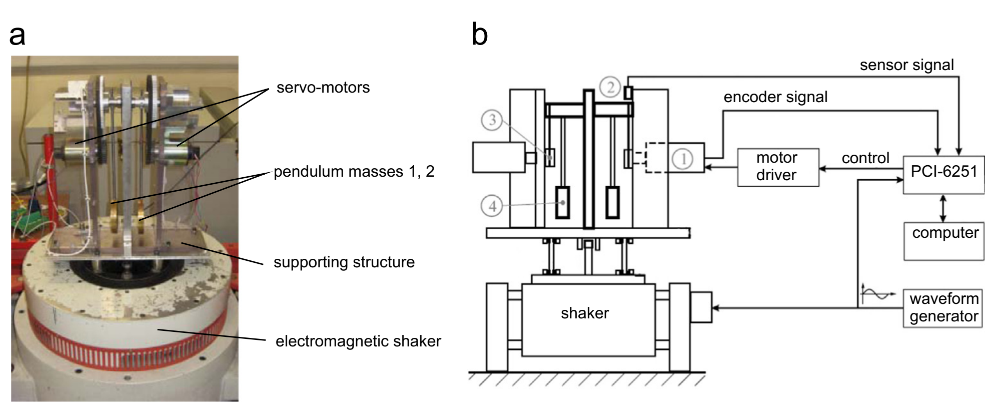

Fig. 12. (a) Photography of the experimental rig and (b) block diagram of the measurements set-up.

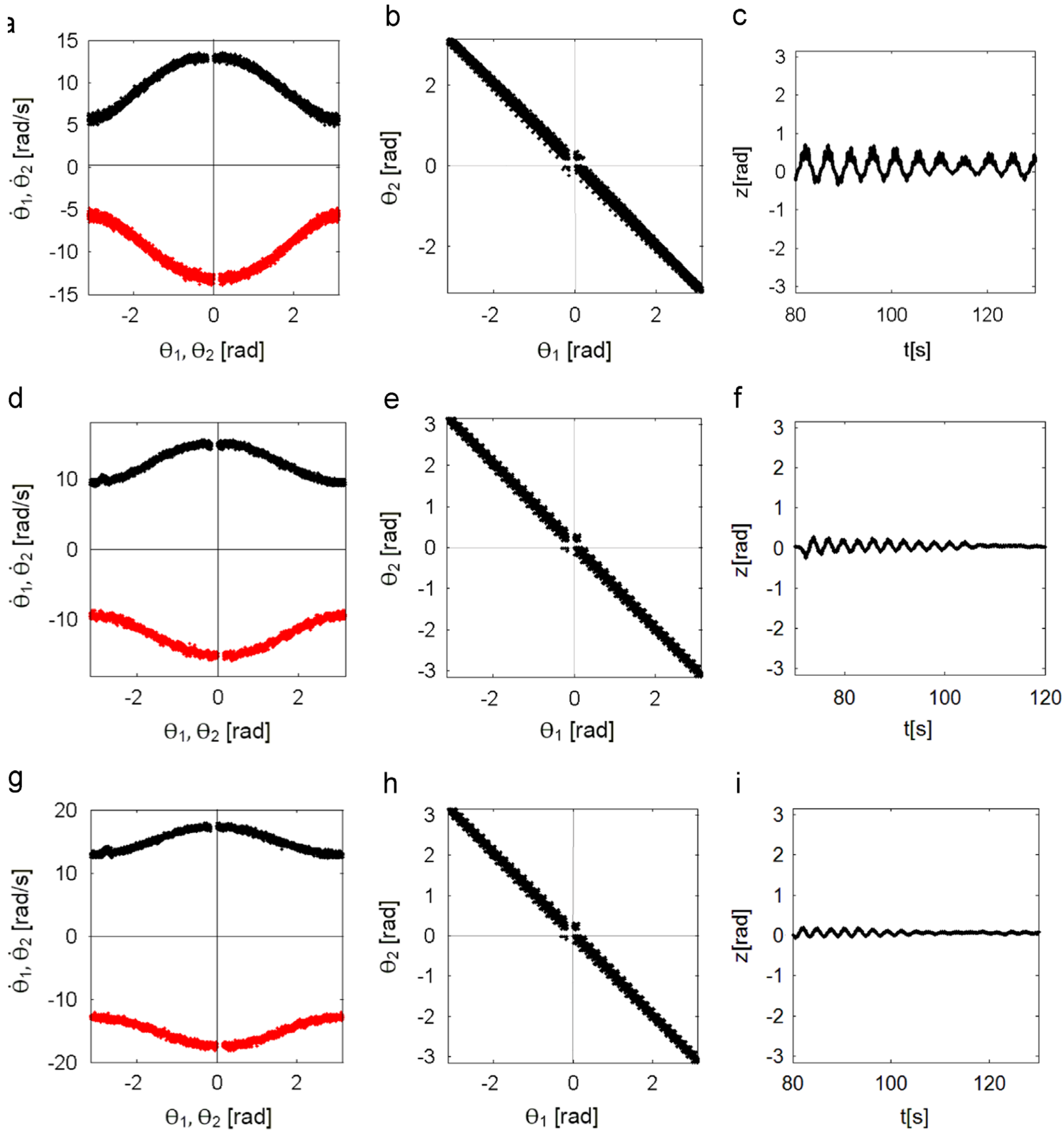

Fig. 13. Synchronized rotation for harmonic excitation from the shaker (a)–(c) $f=1.5$ Hz, $A=0.012$ m, (d)–(f) $f=2$ Hz, $A=0.012$ m, (g)–(i) $f=2.5$ Hz, $A=0.009$ m.

coupling strength of the structure is low, the system exhibits a natural tendency to synchronize. Results showing different types of synchronized solutions have been presented. Only two phase configurations of synchronized rotational motion are possible for the considered system of identical pendulums: complete synchronization in phase or in antiphase.

The difference between the synchronization in phase and in antiphase between the two rotating pendulums has been discussed. It has been shown that the latter one has a property of damping lateral vibrations of the supporting structure. It represents a more desirable response as it maintains the system at a lower energy level. Consequently less energy is required to initiate and maintain a synchronized in antiphase rotation than for the rotation in phase or rotation of a single pendulum, while the other one remains fixed. It has been shown that the type of synchronization influences the stability of rotational motion by shifting the bifurcation curves. The lower limit of rotational motion is of a particular interest when considering energy harvesting from the system. When pendulums require less energy input to sustain rotation, the energy excess can be extracted. This effect is more visible as the stiffness of the supporting structure decreases, and mutual interactions between the pendulums and the structure enhance. By studying the response of the base, it has been shown that the rotating pendulum excites the base to oscillate, unless two pendulums rotate in antiphase.

It has been demonstrated that the synchronized state can be preserved also in the presence of noise in experimental conditions or when the pendulums considered are not identical. For the non-identical case no complete synchronization is possible, instead phase locking and phase synchronization occur, with the phase shift oscillating with an amplitude dependent on the value of the parameter mismatch. The critical value of the parameter mismatch has been identified, beyond which the responses become asynchronous. The influence of the parameter mismatch on the synchronization has been more visible in idealized numerical conditions than in noisy experimental conditions, where

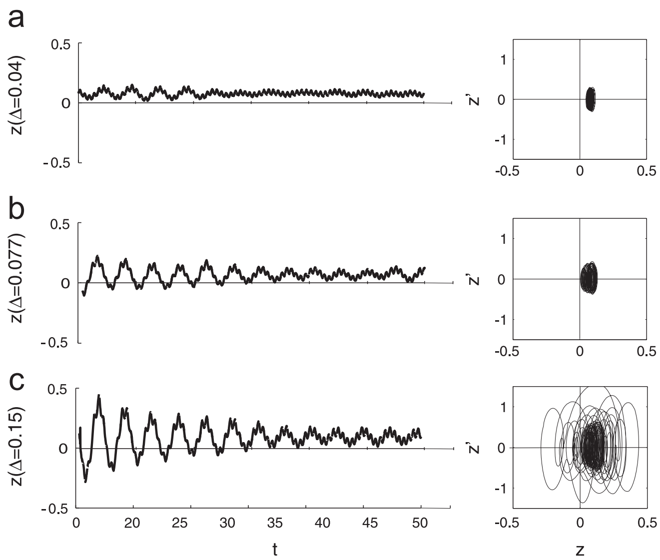

Fig. 14. Experimental time histories of phase variable z and corresponding phase planes for $f=2.5$ Hz, $A=0.012$ m and different values of parameter mismatch, $\Delta=0.04$, $\Delta=0.077$, $\Delta=0.15$.

natural imperfections are always present and system demonstrates imperfect complete synchronization.

Acknowledgements

Tomasz Kapitaniak has been supported by the Polish Department for Scientific Research (DBN) under Project no. N N501 249238. Anna Najdecka would like to acknowledge a PhD studentship from the Northern Research Partnership.

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