Unstable as a Pendulum

TOMASZ KAPITANIAK
Division of Dynamics,
Technical University of Łódź,
Committee on Mechanics,
Polish Academy of Sciences
tomaszka@p.lodz.pl

Can we predict the behavior of evolving systems? While it is sometimes easy to do so, as in the case of an ordinary, slightly tilted pendulum, there are some systems whose ultimate state is practically impossible to ascertain

Practical systems of any sort have a set of desired working conditions, and so while designing an engineering device one typically assumes that it will operate in such conditions. Unfortunately, when a system is nonlinear this state cannot always be guaranteed and one must bear in mind that undesired working conditions might damage the device. Such a situation may occur while modeling biological or geophysical systems, for instance. One may find that the system under consideration operates in a number of different states with different meanings – like for example life and death in biological systems or good or bad weather in geophysical ones. Such situations are usually modeled and tackled using the mathematical concept of *attractors*.

The notion of attractor is the fundamental concept in the theory of dynamical systems. Consider the dynamical system $dx/dt = f(x)$ where $f(x)$ is a function which fulfils all the conditions necessary for the above equation to have a unique solution, x belonging to R^n – this n-dimensional real space is called a phase space of the equation. The minimal subset of R^n, A, with the property that $x(t) \to A$ as $t \to \infty$, is called an attractor. Typical attractors are fixed points (equilibria), limit cycles (periodic behavior), tori (quasiperiodic behavior) and strange attractors (chaotic behavior).

One of the typical features of a nonlinear system is the existence of co-existing attractors, i.e. for a given set of parameter values, depending on initial conditions, the system may move toward a different attractor. This feature is called multistability. To understand the dynamical behavior of such systems it is necessary to calculate the basin of attraction for each coexisting attractor. In a number of cases the structure of the basins and their bifurcations leads to unexpected dynamical uncertainty; *a priori* one cannot predict which attractor the system will evolve based on. Some of these cases are described in this paper.

One of the simplest mechanical systems with more that one possible attractor is the inverted pendulum. As shown below, three equilibrum positions A, B and C are possible. Positions A and C are attractors and B is an unstable equilibrium. The attraction basins of attractors A and C are shown in blue and yellow, respectively. The basin boundaries are well-defined as straight lines. Assume that the initial conditions can be set with a precision ε, so if the initial conditions are not within the ε-wide bands around the boundaries one can easily predict towards which attractor the system will go.

A more complicated case occurs when the basins' boundary has a fractal structure. An example of such a case is to be found in the dynamics of the an externally excited pendulum as seen on Fig. 3. There exist excitations for which the pendulum performs

Tomasz Kapitaniak, Professor of Mechanical Engineering, Dynamics, and Chaos, researches a wide range of different subjects of nonlinear dynamics, such as: chaos, stochastic mechanics, nonlinear vibrations, and the synchronization of systems

T. Kapitaniak

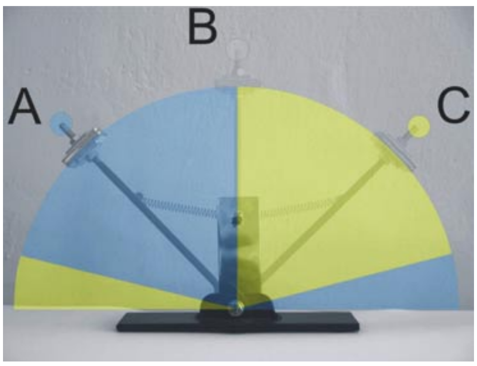
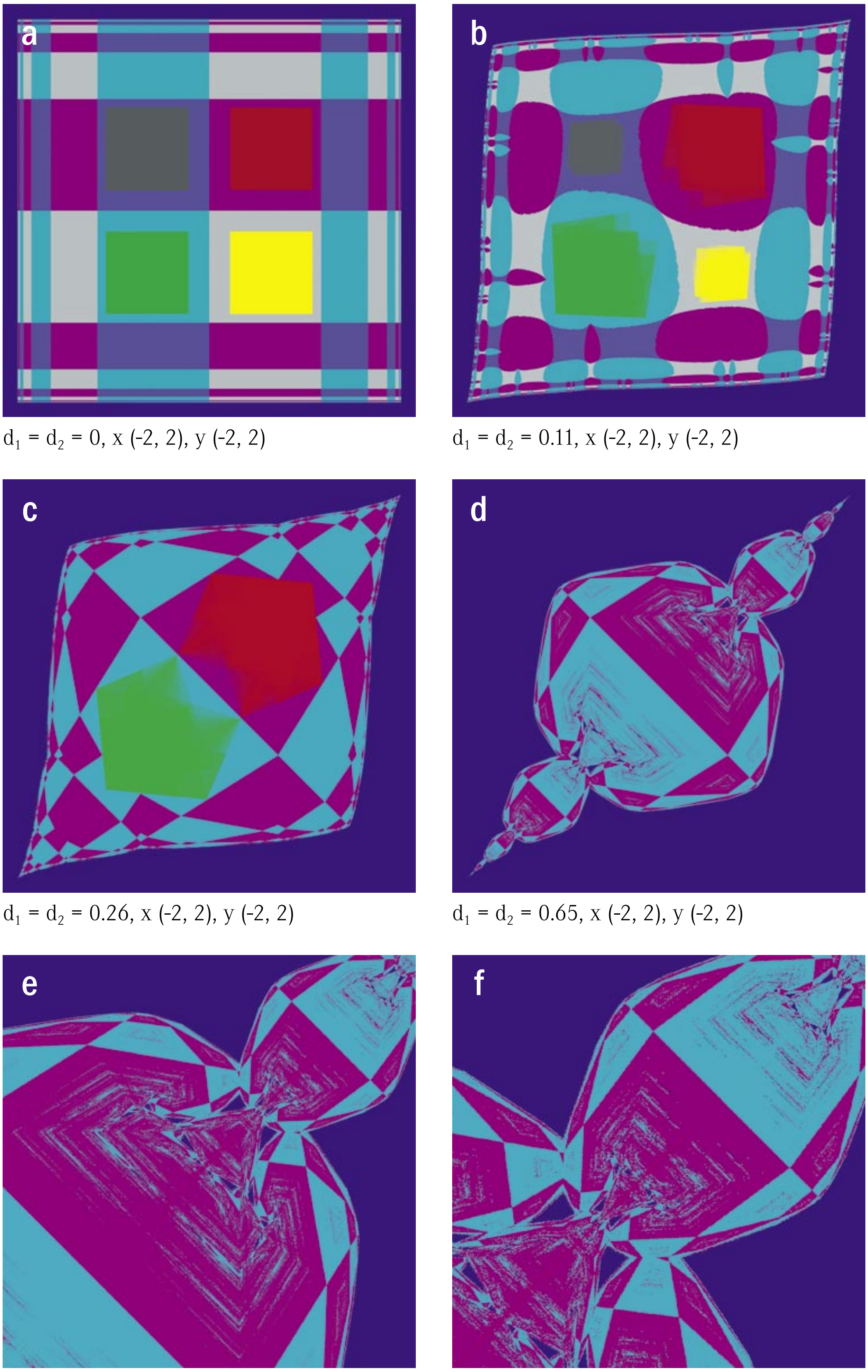


Figure 1: An inverted pendulum and the attraction basins of its attractors: if the initial state is in the blue areas, the system will eventually come to rest at point A (initial states in the yellow areas tend to develop towards point C)

clockwise and counterclockwise rotation. These two periodic attractors are represented respectively as A and B and their basins of attraction are shown in purple and light blue, respectively. The basin boundary here has a fractal structure. Large domains of



T. Kapitaniak

$d_1 = d_2 = 0$, x (-2, 2), y (-2, 2)

$d_1 = d_2 = 0.11$, x (-2, 2), y (-2, 2)

$d_1 = d_2 = 0.26$, x (-2, 2), y (-2, 2)

$d_1 = d_2 = 0.65$, x (-2, 2), y (-2, 2)

$d_1 = d_2 = 0.65$, x (-0.5, 1.5), y (-0.5, 1.5)

$d_1 = d_2 = 0.65$, x (0.5, 1.5), y (0.5, 1.5)

Figure 2: the basins of attraction of the attractors for the dynamic system of equations shown in the text. The four symmetrical chaotic attractors initially seen in (a) as dark grey, green, yellow, and red regions, are preserved at a small coupling (b), but disappear when the coupling increases (c,d). The attractors far from the main diagonal x = y are destroyed first. If the system is evolving towards one of the attractors that subsequently becomes destroyed, it is impossible to predict to which of the remaining attractors the system trajectory will switch

the phase space have the property that in any neighborhood of a given point which belongs to the basin of one attractor, there exist points which belong to the basin of the other attractor. In such domains one cannot predict the fate of the system trajectory only knowing initial conditions with ε uncertainty, although there are certain domains, for example the neighborhoods of the attractors, in which such prediction is indeed possible.

The worst predictability is seen in dynamical systems which can actually change attractors, as a result of the destruction of the original attractors or through small (theoretically infinitely small) external perturbation. These cases will be described using the example of the following discrete dynamical system (a two-dimensional piecewise linear map):

$$x_{n+1} = px_n + l/2\,(1 - p/l)\,(|x_n + 1/l| - |x_n - 1/l|) + d_1\,(y_n - x_n)$$
$$y_{n+1} = py_n + l/2\,(1 - p/l)\,(|y_n + 1/l| - |y_n - 1/l|) + d_2\,(x_n - y_n)$$

For $d_{12} = 0$, $F_{l,p}(0)$ has four symmetrical chaotic attractors $A_{(i)}$, i = 1; 4 inside I x I, where I = [2; 2]. These attractors shown together with their basins of attraction in Figure 2(a). The basins of attractors A_1 (dark grey), A_2 (red), A_3 (green) and A_4 (yellow) are shown in dark blue, purple, light blue, and light grey, respectively while the basin of attraction of infinity is shown in navy blue.

As the computer experiment presented in Figure 2b-d shows, such types of attractors are preserved at a small coupling $|d_{1,2}| << 1$ (Figure 2b), but they disappear when the coupling increases (Figure 2c,d). First the attractors far from the main diagonal x = y, i.e. $A_{(2)}$ and $A_{(4)}$, are destroyed (Figure 2c).

Assume that a dynamical system is evolving on one of its attractors, but this attractor becomes destroyed. It is then impossible to predict to which of the remaining attractors the system trajectory will switch. The destruction of one attractor when at least two other attractors remain is called a multiple choice bifurcation, which is a source of dynamical unpredictability.

In Figure 2d we observe that x = y and two-dimensional attractors are reduced to two symmetrical one-dimensional attractors at the main diagonal x = y. As can be seen in the enlargements shown in Figure 2e-f, in any neighborhood of attractor A (or B) there are points which belong to the basin of another attractor B (or A). In such a case the basin of A (B) is riddled by the basin of B (or A). The riddled basins give another possible dynamical uncertainty, as a system trajectory evolving on one attractor can switch attractors as a result of small external perturbation.

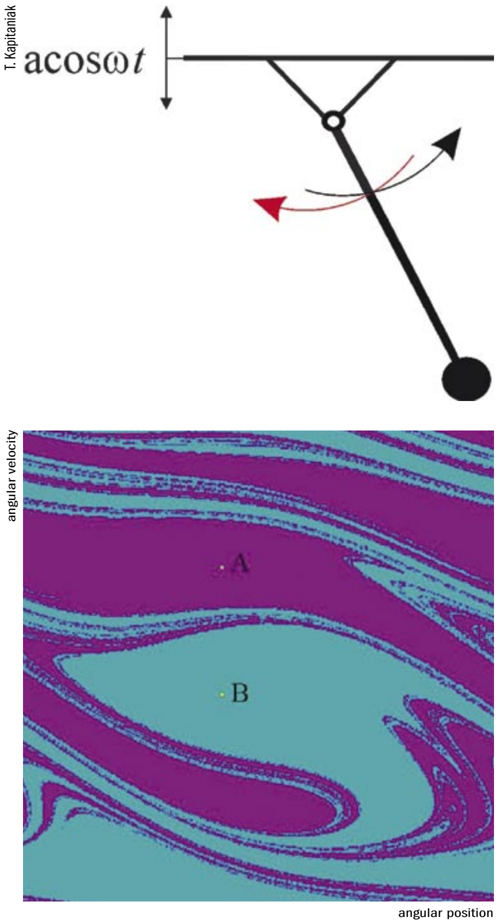


Figure 3: the setup of an externally excited pendulum (above) and the attraction basins of its attractors in the space of initial states (below)

To recapitulate, multistability has been found to be common in dynamical systems – mechanical systems with impacts and dry friction, electrical nonlinear circuits, biological and economical models being typical examples. In such systems one can expect to find one of the dynamical uncertainties described here. Particularly in systems with noise, such uncertainties can lead to unexpected phenomena, in most cases with dramatic results. Dynamical uncertainties are currently a main topic of extensive worldwide research. ■

Further reading:

Nusse H.E., Yorke J.A. (1994). *Dynamics: Numerical Explorations*, New York: Springer Verlag.

Kapitaniak, T, (2000). *Chaos for Engineers: Theory, Applications and Control*, New York: Springer Verlag.